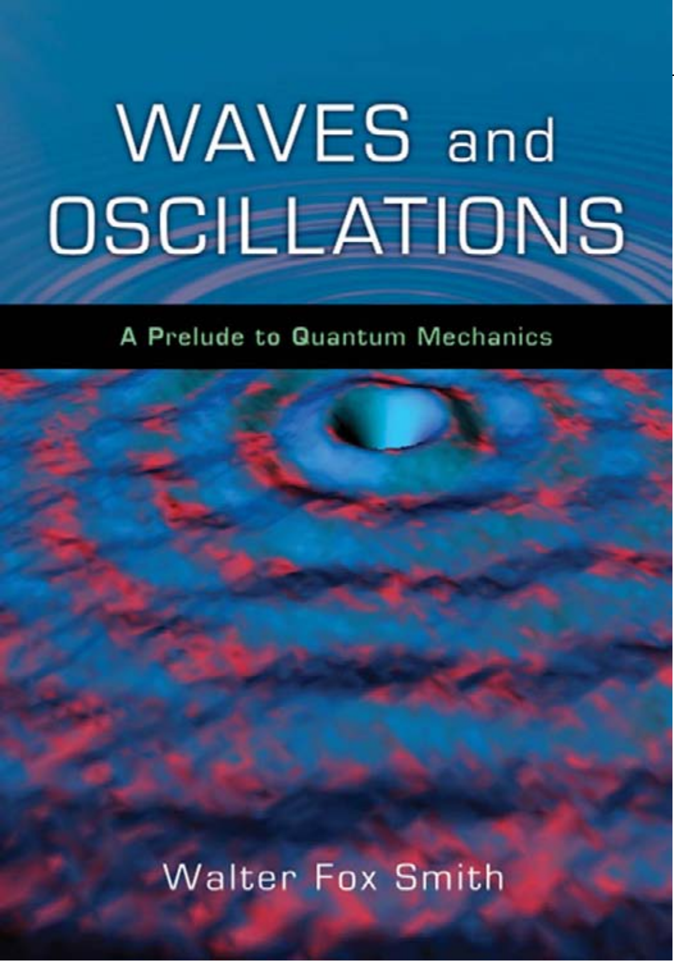
WAVES and
OSCILLATIONS
A Prelude to Quantum Mechanics
Walter Fox Smith

Waves and Oscillations

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# Waves and Oscillations

## A Prelude to Quantum Mechanics

Walter Fox Smith

OXFORD
UNIVERSITY PRESS
2010

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Published by Oxford University Press, Inc.
198 Madison Avenue, New York, New York 10016
www.oup.com



*Library of Congress Cataloging-in-Publication Data*
Smith, Walter Fox.
Waves and oscillations : a prelude to quantum mechanics /
Walter Fox Smith.
p. cm.
Includes index.
ISBN 978-0-19-539349-1
1. Wave equation. 2. Mathematical physics. I. Title.
QC174.26.W28S55 2010
530.12'4–dc22 2009028586

9 8 7 6 5 4 3 2 1

Printed in the United States of America
on acid-free paper

*This book is dedicated to my mother, Barbara Leavell Smith,*
*and to my wife, Marian McKenzie*

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# Preface

## To the student

I wrote this book because I was frustrated by the other textbooks on this subject. Waves and oscillations are enormously important for current research, yet other books don't stress these connections. The ideas and techniques that you will learn from this book are *exactly* what you need to be ready for a study of quantum mechanics. Every physics professor understands this linkage, and yet other books fail to emphasize it, and often use notations which are different from those used in quantum mechanics. Other books make little effort to keep you engaged. I can't teach you by myself, nor can your professor; you have to learn, and to do this you must be active. In this book, I've provided tools so that you can assess your learning as you go; these are described immediately after the table of contents. Use them. Read with paper and pencil handy. As a scientist, you know that only by understanding the assumptions made and the details of the derivations can you have your own logical sense of how it all fits together into a self-consistent whole. Visit this book's website. There, you will find links to current physics, chemistry, biology, and engineering research that is related to the topics in each chapter, as well as lots of other stuff, some purely fun and some purely educational (but most of it both). Hopefully, there will be a second edition of this book in the future; if you have suggestions for it, please e-mail me: wsmith@haverford.edu.

## To the instructor

Please visit the website of this book. You'll find materials in the website that will make your life easier, including full solutions and important additional support materials for the end-of-chapter problems, lecture notes which complement the text (including additional conceptual questions, worked examples, applications to current research and everyday life, animations, and figures), as well as custom-developed interactive applets, video and audio recordings, and much more. The following sections can be omitted without affecting comprehension of later material: 1.10, 1.12, 2.3–2.6, 3.5–3.6, 4.5, 4.7–4.8, 6.6–6.7, 8.6–8.7, 9.9, 9.11, 10.8–10.9, and Appendix A. If necessary, one can skip all of chapter 6, except for the part of section 6.5 starting with the "Core example" through the end of the section; however omitting the rest of chapter 6 means that the students won't be exposed to any matrix math or to the idea of an eigenvalue equation. (They are exposed copiously to eigenvectors and eigenfunctions in other chapters, but the word "eigenvalue" is used only in chapter 6.) If you have questions or comments, please contact me: wsmith@haverford.edu.

## Acknowledgments

This book builds on the enormous efforts of my predecessors. Like any textbook author, I have consulted many dozens of other works in developing my presentation. However, three stand out as particularly helpful: *Vibrations and Waves*, by A. P. French (Norton 1971), *The Physics of Vibrations and Waves, 6th Ed.*, by H. J. Pain (Wiley 2005), and *The Physics of Waves*, by H. Georgi (Prentice-Hall, 1993).

I am deeply grateful to my physics colleague Peter J. Love, who cheerfully answered endless questions from me, taught from draft versions of the book and gave me essential feedback, and made key suggestions for several sections. I am also most thankful to my other colleagues in physics who supported me in this effort and answered my many questions: Jerry P. Gollub, Suzanne Amador-Kane, Lyle D. Roelofs, and Stephon H. Alexander. I also received very valuable inputs from colleagues in math, particularly Robert S. Manning, and chemistry, including Casey H. Londergan, Alexander Norquist, and Joshua A. Schrier. I also thank Jeff Urbach of Georgetown University and Juan R. Burciaga of Lafayette College who used draft versions of the text in their courses, and provided helpful feedback.

I am profoundly thankful for the proof-reading efforts, and suggested edits and end-of-chapter problems from Megan E. Bedell, Martin A. Blood-Forsythe, Alexander D. Cahill, Wesley W. Chu, Donato R. Cianci, Eleanor M. Huber, Anna M. Klales, Anna K. Pancoast, Daphne H. Paparis, Annie K. Preston, and Katherine L. Van Aken. Special thanks are due to Andrew P. Sturner for his tireless efforts and suggestions, right up to the last minute.

Finally, I am most deeply grateful to my family, for their support and encouragement throughout the writing of this book. My children Grace, Charlie, and Tom checked up on my progress every day, and suggested things in everyday life connected to waves and oscillations. My good friend Michael K. McCutchan gave deep proofreading and editing help, and support of all kinds throughout. Finally, words cannot express my gratitude for the efforts of my wife, Marian McKenzie, who did almost all the computerizing of figures, helped with editing, and provided the much-needed emotional support. This book would never have been published without her encouragement.

# Learning Tools Used in This Book

Throughout this text you will find a number of special tools which are designed to help you understand the material more quickly and deeply. Please spend a few moments to read about them now.

### Concept test

This checks your understanding of the ideas in the preceding material.

### Self-test

Similar to a concept test, but more quantitative. It will require a little work with pencil and paper.

### Core example

Unlike an ordinary example, these are not simply applications of the material just presented, but rather are an integral part of the main presentation. There are some topics that are much easier to understand when presented in terms of a specific example, rather than in more abstract general terms.

### Your turn

In these sections, you are asked to work through an important part of the main presentation. Be sure to complete this work before reading further.

## Concept and skill inventory

At the end of each chapter, you'll find a list of the key ideas that you should understand after reading the chapter, and also a list of the specific skills you should be ready to practice.

# Contents

# Waves and Oscillations

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# 1 Simple Harmonic Motion

All around us, sinusoidal waves astound us!
From "The Waves and Oscillations Syllabus Song," by Walter F. Smith

## 1.1 Sinusoidal oscillations are everywhere

You are sitting on a chair, or a couch, or a bed, something that is more or less solid. Therefore, every atom within it has a well-defined position. However, if you could look very closely, you'd see that *every one* of those atoms right now is vibrating relative to this assigned position. The hotter your chair the more violent the vibration, but even if your chair were at absolute zero, every atom would *still* be vibrating! Of course, the same is true for every atom in every solid object throughout the universe—right now, each one of them is vibrating relative to its assigned or "equilibrium" position within the solid.

The vibration of a particular one of these atoms might follow the pattern shown in the top part of figure 1.1.1. The pattern appears complicated, but we will show in the course of this book that it is really just a summation of simple sinusoids (as shown in the lower part of the figure), each of which is associated with a "normal mode" of the solid that contains the atoms. (Over the next several chapters, we'll explore what the term "normal mode" means.)

The complexity shown in the top part of the figure arises because the solid has many "degrees of freedom"; every one of the atoms in the solid can move in three dimensions, and each atom is affected by the motion of its neighbors. The approach of physics, and it has been enormously successful in an astonishing variety of situations, is to build up an understanding of complex systems through a thorough understanding of simplified versions. For example, when studying trajectories, we begin with objects falling straight down in a vacuum, and gradually build up to an understanding of three-dimensional trajectories, including effects of air resistance and perhaps tumbling of the object.

So, to understand the motion of the atom, we begin with systems that have only one degree of freedom, that is, systems that can only move in one direction and moreover don't have neighbors that move. A good example is a tree branch. If you pull it straight up and then let go, the resulting motion looks roughly as shown in figure 1.1.2. Again, we see a sinusoidal motion, although in this case it is "damped," meaning that over

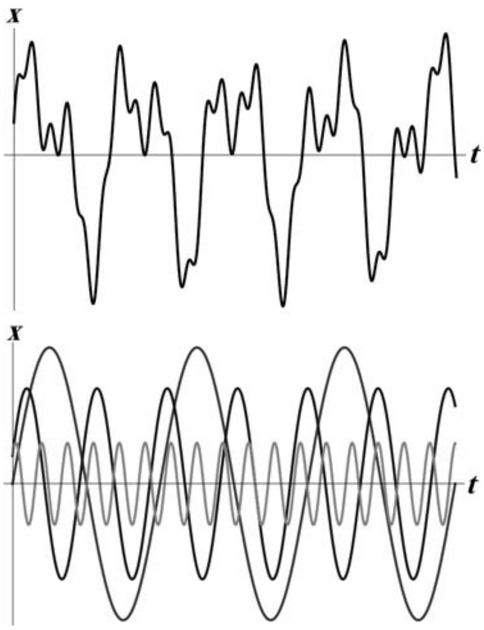


**Figure 1.1.1** Top: motion of an atom in a solid. Bottom: Sine waves that, when added together, create the waveform shown in the top part.

time the motion decays away. Hold a pen or a pencil loosely at one end with your thumb and forefinger, with the rest of the pencil hanging below. Push the bottom of the pencil to one side, and then let go—the resulting motion looks similar to figure 1.1.2, though this time the quantity being plotted is the angle of the pencil relative to vertical.

In fact, if you take *any* object that is in an equilibrium position, displace it from equilibrium, and then let go, you'll get this same type of damped sinusoidal response, as we will show quite easily in section 1.2. This type of oscillation is enormously important, not only in the macroscopic motion of objects, machine parts, and so on but also, perhaps surprisingly, in the performance of many electronic circuits, as well as in

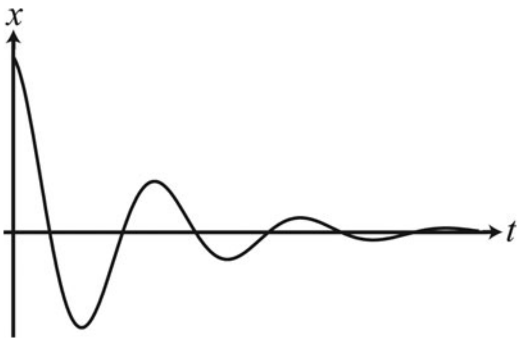


**Figure 1.1.2** Motion of a tree branch when pulled up and then released.

the detailed understanding of the motions of atoms and molecules, and their interaction with light.

So, sinusoidal motion really is all around us, and something which any scientist must understand deeply. However, there is another perhaps even more important reason to study oscillations and waves: the mathematical tools and intuition you will develop during this study are *exactly* what you need for quantum mechanics! This is not surprising, since much of quantum mechanics deals with the study of the "wave function" which describes the wave nature of objects such as the electron. However, the connection of the field of waves and oscillations to that of quantum mechanics is much deeper, as you'll appreciate later. For now, rest assured that you are laying a very solid foundation for your later study of quantum mechanics, which is the most important and exciting realm of current physics research and application.

## 1.2 The physics and mathematics behind simple sinusoidal motion

To start our quantitative study, we follow the approach of physics and consider the simplest possible system: one with no damping. This means that all the forces acting on the object are conservative and so can be associated with a potential energy.

A body in stable equilibrium is, by definition, at a local minimum of the potential energy *versus* position curve, as shown in figure 1.2.1. For convenience, we choose $x = 0$ at the equilibrium position. Except in pathological cases, the potential energy function $U(x)$ near $x = 0$ can be approximated by a parabola, as shown. We write this parabolic or "harmonic" approximation in the form $U(x) \approx \frac{1}{2}kx^2 + \text{const.}$ for reasons that will become apparent in the next sentence.

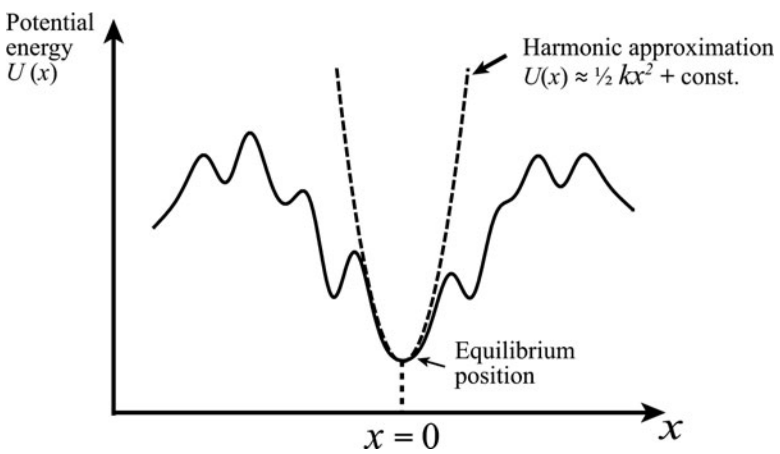


**Figure 1.2.1** The Harmonic Approximation, valid for small vibrations around equilibrium.

The force acting on the body can then be found using $F = -\dfrac{dU}{dx} = -kx$. The relation

$$\boxed{\boxed{F = -kx}} \tag{1.2.1}$$

is known as “Hooke’s Law,” after its discoverer Robert Hooke (1635–1703).[1] The quantity $k$ is called the “spring constant.” To find the position of the body as a function of time, $x(t)$, we will follow a three-step procedure. We’ll use the same procedure throughout the book, for progressively more complex systems. To save space, we simply write $x$ remembering that this is shorthand for the function $x(t)$.

**1. Write down Newton’s second law for each of the bodies involved.**
In this case, there is only one body, so we have

$$\left.\begin{aligned} F = ma = m\frac{d^2x}{dt^2} \\ F = -kx \end{aligned}\right\} \Rightarrow m\frac{d^2x}{dt^2} = -kx. \tag{1.2.2}$$

This is a “differential equation” or DEQ which simply means that it is an equation that involves a derivative. (If you haven’t had a course in DEQs, don’t worry; we’ll go through everything you need to know for this course and for a first course in quantum mechanics.) This is called a “second order DEQ,” because it contains a second derivative. The “solution” for this equation is a function $x(t)$ for which the equation holds true—in this case, a function for which, when you take two time derivatives and multiply by $m$ (as indicated on the left side of the equation), then you get back the same function times $-k$ (as indicated on the right side of the equation). This is the solution that we are trying to find, since it tells us the position of the object at all times. One important thing to know right away is that there is *no general recipe* for finding the solution that works for all second-order DEQs. However, for many of the most important such equations in physics, we can *guess* a solution based on our intuition and then *check* to determine whether our guess is really right, as shown in the following steps.

1. Some scholars feel that Robert Hooke is one of the most underappreciated figures in science. He was the founder of microscopic biology (he coined the word “cell”), he discovered the red spot on Jupiter and observed its rotation, he was the first to observe Brownian motion (150 years before Brown), and discovered Uranus 108 years before the more-publicized discovery by Herschel. Unfortunately, it seems that Hooke spread himself too thin, and never got around to publishing many of his results. Hooke and Newton, though originally on friendly terms, later became fierce rivals. It appears that Hooke conceptualized the inverse square law of gravity and the elliptical motion of planets before Newton, and discussed this idea briefly with Newton. Newton (unlike Hooke) was able to show quantitatively how the inverse square law predicts elliptical orbits, and felt that Hooke was pushing for more recognition than he deserved in this very important discovery. Some scholars feel that, when Newton became the president of the Royal Society (the leading scientific organization of the time in England), he may intentionally have “buried” the work of Hooke, but there is no hard evidence to support this.

To save space, we write $\frac{d^2x}{dt^2}$ as $\ddot{x}$. (Each dot represents a time derivative,[2] so that $\dot{x}$ represents $\frac{dx}{dt}$.) We rearrange equation (1.2.2) slightly to give

$$\ddot{x} = -\frac{k}{m}x. \tag{1.2.3}$$

This is called the "equation of motion."

**2. Using physical intuition, guess a possible solution.**
Observation of a mass bouncing on a spring suggests that its motion may be sinusoidal. The most general possible sinusoid can be expressed as

$$x = A\cos(\omega t + \varphi) \tag{1.2.4}$$

The values of the "adjustable constants" $A$ and $\varphi$ depend on the initial conditions, as we will discuss later.

**3. Plug the guess back into the system of DEQs to see if it is actually a solution, and to determine whether there are any restrictions on the parameters that appear in the guess.**
In this case, the "system of DEQs" is the single equation (1.2.3). Before you look at the next paragraph, plug the guess (1.2.4) into (1.2.3), verify that it is indeed a solution, and find what the "parameter" $\omega$ must be in terms of $k$ and $m$.

You should have found that

$$\omega = \sqrt{k/m} \tag{1.2.5}$$

So, we see that sinusoidal vibration, also known as "simple harmonic motion" or SHM, is universally observed for vibrations that are small enough to use the Harmonic Approximation shown in figure 1.2.1.

As described in section 1.3, $\omega$ equals $2\pi$ times the frequency of the motion and is called the "angular frequency."

## 1.3 Important parameters and adjustable constants of simple harmonic motion

Figure 1.3.1 shows a graph of the SHM represented by equation (1.2.4). Any such sinusoidal motion can be described with three quantities:

1. The amplitude $A$. As shown, the maximum value of $x$ is $A$, and the minimum value is $-A$.

2. The dot notation was invented by Isaac Newton. It is very convenient for us, because we have to deal with time derivatives so frequently. However, it is generally felt that, because historical English mathematicians continued to use this notation so long, they were held back relative to their German counterparts, who used Gottfried Leibniz's d/d$t$ notation instead. (Leibniz's notation is more flexible, and we will use it where convenient.)

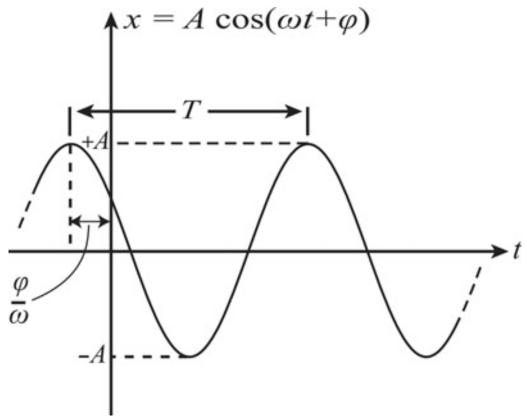


**Figure 1.3.1** Simple harmonic motion of period $T$ and amplitude $A$.

2. The period $T$. This is the time between successive maxima, or equivalently between successive minima. The period is the time needed for one complete cycle, so that when the time $t$ changes by $T$, the argument of the cosine in $x = A\cos(\omega t + \varphi)$ must change by $2\pi$. Therefore,

$$\omega(t+T)+\varphi = \omega t + \varphi + 2\pi,$$

so that

$$\boxed{\boxed{T = 2\pi/\omega}} \tag{1.3.1}$$

(This equation is shown with a double border because we'll be referring to it so frequently. Equations shown this way are so very important that you will find it helpful to begin memorizing them right away.) The frequency $f$ is given by $1/T$, so that

$$\boxed{\boxed{\omega = 2\pi f}} \tag{1.3.2}$$

For this reason, $\omega$ is called the "angular frequency." We will use it *continually* for the rest of the text, so get accustomed to it now! We will encounter various different angular frequencies later, so we give the special name $\omega_0$ to the angular frequency of simple harmonic motion, that is,[3]

$$\boxed{\boxed{\omega_0 \equiv \sqrt{k/m}}} \tag{1.3.3}$$

(Note: the "0" subscript here does not indicate a connection to $t = 0$, but it is universally used.)

3. The "initial phase" $\varphi$. The position at $t = 0$ is determined by a combination of $A$ and $\varphi$. It is easy to find the relation between these two "adjustable constants"

3. Physicists use the symbol "$\equiv$" to mean "is defined to be."

on one hand and the initial position $x_0$ and the initial velocity $v_0$ on the other. From equation (1.2.4): $x = A\cos(\omega t + \varphi)$ we obtain:

$$x_0 = A\cos\varphi \quad \text{and} \quad v_0 = \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = -\omega_0 A \sin\varphi$$

**Your turn:** From these, you should now show that

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \quad \text{(1.3.4a)} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{-v_0}{\omega_0 x_0}\right). \text{ (1.3.4b)}$$

(We use the term "parameter" to refer to a quantity determined by the physical properties of a system, such as mass, spring constant, or viscosity. Thus, $\omega_0$ is a parameter. In contrast, we use "adjustable constant" to designate a quantity that is determined by initial conditions. Thus, $A$ and $\varphi$ are adjustable constants.)

As mentioned earlier, the equation of motion (1.2.3) is a second-order DEQ, because the highest derivative is of second order. It can be shown that the most general solution to a second-order DEQ contains two (and no more than two) adjustable constants.[4] (We know that this must be true for our case, since we need to be able to take into account (1) the initial position and (2) the initial velocity when writing out a particular solution, therefore we need to be able to adjust two constants.) So, we can be confident that equation (1.2.4): $x = A\cos(\omega t + \varphi)$ is the *general* solution to equation (1.2.3): $\ddot{x} = -\dfrac{k}{m}x$. An example of a nongeneral solution would be $x = A\sin\omega_0 t$; you should verify that this satisfies equation (1.2.3). But this is the same as equation (1.2.4), with the particular choice $\varphi = -\pi/2$.

Look again at equation (1.3.3): $\omega_0 = \sqrt{k/m}$. There is something about it that is absolutely astonishing. *The angular frequency depends only on the spring constant and the mass – it doesn't depend on the amplitude!* It would be very reasonable to expect that, for a larger amplitude, it would take longer for the system to complete a cycle, since the mass has to move through a larger distance. However, at larger amplitudes the restoring force is larger and this provides exactly enough additional acceleration to make the period (and so $\omega$) constant. The fact that the frequency is independent of amplitude is critical to many applications of oscillators, from grandfather clocks to radios to microwave ovens to computers. Most of these do not actually have separate masses and springs inside them, but instead have combinations of components which are described by exactly analogous DEQs, and so exhibit exactly analogous behavior. We'll explore many of these in chapter 2, but we start now with the two most basic, and most important, examples.

4. For the special case of a "linear" (meaning no terms such as $x^2$ or $x\dot{x}$), "homogeneous" (meaning no constant term) DEQ, such as equation (1.2.3), this theorem is often phrased in the alternate form, "The general solution of a linear, homogeneous second-order DEQ is the sum of two independent solutions." An example for our case would be $x = A_1\cos\omega_0 t + A_2\sin\omega_0 t$. However, you can easily show (see problem 1.7) that this can be expressed in the form $x = A\cos(\omega_0 t + \varphi)$, with $A = \sqrt{A_1^2 + A_2^2}$ and $\varphi = \tan^{-1}(-A_2/A_1)$.