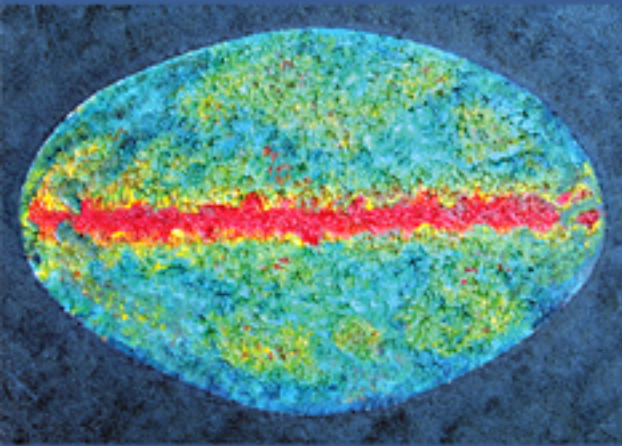


RUTH DURRER

THE Cosmic Microwave Background



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THE COSMIC MICROWAVE BACKGROUND

The cosmic microwave background (CMB) is the radiation left over from the big bang. Recent analysis of the fluctuations in this radiation has given us valuable insights into our Universe and its parameters.

This textbook examines the theory of CMB and its recent progress. It starts with a brief introduction to modern cosmology and its main successes, followed by a thorough derivation of cosmological perturbation theory. It then explores the generation of initial fluctuations by inflation. In the following chapters the Boltzmann equation, which governs the evolution of CMB anisotropies, and polarization are derived using the total angular momentum method. Cosmological parameter estimation is discussed in detail. The lensing of CMB fluctuations and spectral distortions are also treated.

The book is the first to contain a full derivation of the theory of CMB anisotropies and polarization. Ideal for graduate students and researchers in this field, the textbook includes end-of-chapter exercises, and solutions to selected exercises are provided.

Ruth Durrer is Professor of Theoretical Physics at the Université de Genève. Her research focuses on the cosmic microwave background, cosmic magnetic fields and braneworld cosmology.

THE COSMIC MICROWAVE BACKGROUND

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To Martin, Florian, Melchior and Anna

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Preface

Cosmology, the quest concerning the Universe as a whole, has been a primary interest of human study since the beginnings of mankind. For a long time our ideas about the Universe were dominated by religious beliefs – tales of creation. Only since the advent of general relativity in 1915 have we had a scientific theory at hand that might be capable of describing the Universe. Soon after Einstein's first attempt of a static universe, Hubble and collaborators (Hubble, 1929) discovered that the observable Universe is expanding. This together with the discovery of the cosmic microwave background (CMB) by Penzias and Wilson (Nobel prize 1978) has established the theory of an expanding and cooling universe which started in a 'big bang'.

For a long time observations that have led to the determination of cosmological parameters, such as the rate of expansion, the so-called Hubble parameter, the mean matter density of the Universe, or its curvature, have been very sparse and we could only determine the order of magnitude of these parameters.

During the last decade this situation has changed significantly and cosmology has entered an era of precision measurements. This major breakthrough is to a large extent due to precise measurement and analysis of the CMB. In this book I develop the theory which is used to analyse and understand measurements of the CMB, especially of its anisotropies and polarization, but also its frequency spectrum. The Nobel prize was awarded to George Smoot and John Mather, in 2006, for the discovery of these anisotropies and for precise measurements of the CMB spectrum.

The book is directed mainly towards graduate students and researchers who want to obtain an overview of the main developments in CMB physics, and who want to understand the state-of-the-art techniques which are used to analyse CMB data. I believe that the theory of CMB physics is now sufficiently mature for a book on this topic to be useful. I shall not enter into any details concerning CMB experiments. This is by no means because I consider them less interesting, but rather that they

are still in full development and will hopefully make significant progress in the near future. Of course, my background is also that of a theoretical physicist and my main interest lies in the theoretical aspects of CMB physics. I hope, however, that this book will also be useful to CMB experimentalists who want to know what happens inside their cosmic parameter estimation routines.

It is assumed that the reader is familiar with undergraduate physics including the basics of general relativity, and has an elementary knowledge of quantum field theory and particle physics. The beauty of cosmology lies in the fact that it employs more or less all fields of physics starting with general relativity over thermodynamics and statistical physics to electrodynamics, quantum mechanics and particle physics. In this book I do not want to present an introduction to these topics as well since, first of all, there exist wonderful textbooks on all of them and second you have learned them in your undergraduate physics courses.

Before we start, let me sketch the content of the different chapters and give you a guide on how to read this book.

The first chapter is an overview of the homogeneous and isotropic universe. We present and discuss the Friedmann equations, recombination, nucleosynthesis and inflation. Readers familiar with cosmology may skip this chapter or just skim it.

In Chapter 2 we develop cosmological perturbation theory. This is the basics of CMB physics. The main reason why the CMB allows such an accurate determination of cosmological parameters lies in the fact that its anisotropies are small and can be determined within first-order perturbation theory. In Fourier space the linear perturbation equations become a series of ordinary linear differential equations, which can be solved numerically to high precision without any difficulty. We derive the perturbations of Einstein's equations and the energy-momentum conservation equations and solve them for simple but relevant cases. We also discuss the perturbation equation for light-like geodesics. This is sufficient to calculate the CMB anisotropies in the so-called instant recombination approximation. The main physical effects which are missed in such a treatment are Silk damping on small scales and polarization. We then introduce the CMB power spectrum and draw our first conclusions for its dependence on cosmological and primordial parameters. For example, we derive an approximate formula for the position of the acoustic peaks. An experimentalist mainly interested in parameter estimation may jump, after Chapter 2, directly to Chapter 6 and skip the more theoretical parts between.

The third chapter is devoted to the initial condition. There we explain how the unavoidable quantum fluctuations are amplified during an inflationary phase and lead to a nearly scale-invariant spectrum of scalar and tensor perturbations. We also discuss the initial conditions for mixed adiabatic and iso-curvature perturbations.

In Chapter 4 we derive the perturbed Boltzmann equation for CMB photons. After a brief introduction to relativistic kinetic theory, we first derive the Liouville

equation, i.e. the Boltzmann equation without a collision term. We also discuss the connection between the distribution function and the energy–momentum tensor. We then derive the collision term, i.e. the right-hand side of the Boltzmann equation, due to Thomson scattering of photons and electrons. In this first attempt we neglect the polarization dependence of Thomson scattering. The chapter ends with a list of the full system of perturbation equations for a Λ CDM universe.

In Chapter 5 we discuss polarization. Here we derive the total angular momentum method that is perfectly adapted to the problem of CMB anisotropies and polarization, taking into account its symmetry, which allows a decomposition into modes with fixed total angular momentum. The representation theory of the rotation group and the spin weighted spherical harmonics which are extensively used in this chapter are deferred to an appendix. We interpret some results using the flat sky approximation, which is valid on small angular scales.

Chapter 6 is devoted to parameter estimation. We first discuss the physical dependence of CMB anisotropies on cosmological parameters. After a section on CMB data we then treat in some detail statistical methods for CMB data analysis. We discuss especially the Fisher matrix and explain Markov chain Monte Carlo methods. We also address degeneracies, combinations of cosmological parameters on which CMB anisotropies do not, or only very weakly, depend. Because of these degeneracies, cosmological parameter estimation also makes use of other, non-CMB related, observations. We summarize them in a separate section. We finish the chapter with a discussion of ‘sources’, i.e. inhomogeneously distributed contributions to the energy–momentum tensor, such as topological defects, which may also contribute to the CMB anisotropies and thereby affect the estimated cosmological parameters.

In Chapter 7 we treat lensing of CMB anisotropies and polarization. This second-order effect is especially important on small scales but also has to be taken into account for $\ell \gtrsim 500$ if we want to achieve an accuracy of better than 0.5%. We first derive the deflection angle and the lensing power spectrum. Then we discuss lensing of CMB fluctuations and polarization in the flat sky approximation, which is sufficiently accurate for angular harmonics with $\ell \gtrsim 50$. We conclude the chapter with an overview on other second-order effects.

In the final chapter spectral distortions of the CMB are discussed. We first introduce the three relevant collision processes in a universe with photons and non-relativistic electrons: elastic Compton scattering, Bremsstrahlung and double Compton scattering. We derive the corresponding collision terms and Boltzmann equations. For elastic Compton scattering this leads us to the Kompaneets equation for which we present a detailed derivation. We introduce the timescales corresponding to these three collision processes and determine at which redshift a given process freezes – becomes slower than cosmic expansion. Finally, we discuss the

possible generation of a chemical potential in the CMB spectrum and the Sunyaev–Zel’dovich effect.

All chapters are complemented with some exercises at the end.

In the appendices we collect useful constants and formulae, information on special functions and some more technical derivations. The solutions to a selection of exercises are also given in an appendix.

This book has grown out of a graduate course on CMB anisotropies that I have given on several occasions. Thanks are due to the students of these courses, who have motivated me to write it up in the form of a textbook. I am also indebted to many collaborators and colleagues with whom I have discussed various aspects of the book and who have helped me to clarify many issues. Especially I want to mention Chiara Caprini, Martin Kunz, Toni Riotto, Uros Seljak and Norbert Straumann. I am also immensely grateful to students and colleagues who have read parts of the draft and helped me correct numerous typographical errors and other mistakes: Camille Bonvin, Jean-Pierre Eckmann, Alice Gasparini, Sandro Scodeller and others. Of course all the remaining mistakes are entirely my responsibility. Marcus Ruser and Martin Kunz have also helped me with some of the figures. I also wish to thank Susan Staggs who provided me with a most useful dataset of the CMB spectrum.

Ruth Durrer

1

The homogeneous and isotropic universe

Notation

In this book we denote the derivative with respect to physical time by a prime, and the derivative with respect to conformal time by a dot,

$$\tau = \text{physical (cosmic) time} \quad \frac{dX}{d\tau} \equiv X', \quad (1.1)$$

$$t = \text{conformal time} \quad \frac{dX}{dt} \equiv \dot{X}. \quad (1.2)$$

Spatial 3-vectors are denoted by a bold face symbol such as \mathbf{k} or \mathbf{x} whereas four-dimensional spacetime vectors are denoted as $x = (x^\mu)$.

We use the metric signature $(-, +, +, +)$ throughout the book.

The Fourier transform is defined by

$$f(\mathbf{k}) = \int d^3x f(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (1.3)$$

so that

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3k f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (1.4)$$

We use the same letter for $f(\mathbf{x})$ and for its Fourier transform $f(\mathbf{k})$. The spectrum $P_f(k)$ of a statistically homogeneous and isotropic random variable f is given by

$$\langle f(\mathbf{k}) f^*(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_f(k). \quad (1.5)$$

Since it is isotropic, $P_f(k)$ is a function only of the modulus $k = |\mathbf{k}|$. If f is Gaussian, the Dirac delta function implies that different \mathbf{k} 's are uncorrelated.

Throughout this book we use units where the speed of light, c , Planck's constant, \hbar and Boltzmann's constant, k_B are unity, $c = \hbar = k_B = 1$. Length and time therefore have the same units and energy, mass and momentum also have the same units, which are inverse to the unit of length. Temperature has the same units as energy.

We may use cm^{-1} to measure energy, mass, temperature, or eV^{-1} to measure distances or times. We shall use whatever unit is convenient to discuss a given problem. Conversion factors can be found in Appendix 1.

1.1 Homogeneity and isotropy

Modern cosmology is based on the hypothesis that our Universe is to a good approximation homogeneous and isotropic on sufficiently large scales. This relatively bold assumption is often called the ‘cosmological principle’. It is an extension of the Copernican principle stating that not only should our place in the solar system not be a special one, but also that the position of the Milky Way in the Universe should be in no way statistically distinguishable from the position of other galaxies. Furthermore, no direction should be distinguished. The Universe looks statistically the same in all directions. This, together with the hypothesis that the matter density and geometry of the Universe are smooth functions of the position, implies homogeneity and isotropy on sufficiently large scales. Isotropy around each point together with analyticity actually already implies homogeneity of the Universe.¹ A formal proof of this quite intuitive result can be found in [Straumann \(1974\)](#).

But which scale is ‘sufficiently large’? Certainly not the solar system or our galaxy. But also not the size of galaxy clusters. (In cosmology, distances are usually measured in Mpc (Megaparsec). $1 \text{ Mpc} = 3.2615 \times 10^6 \text{ light years} = 3.0856 \times 10^{24} \text{ cm}$ is a typical distance between galaxies, the distance between our neighbour Andromeda and the Milky Way is about 0.7 Mpc. These and other connections between frequently used units can be found in Appendix 1.)

It turns out that the scale at which the *galaxy distribution* becomes homogeneous is difficult to determine. From the analysis of the Sloan Digital Sky Survey (SDSS) it has been concluded that the irregularities in the galaxy density are still on the level of a few per cent on scales of $100 h^{-1} \text{ Mpc}$ ([Hogg et al., 2005](#)). Fortunately, we know that the *geometry* of the Universe shows only small deviations from the homogeneous and isotropic background, already on scales of a few Mpc. The geometry of the Universe can be tested with the peculiar motion of galaxies, with lensing, and in particular with the cosmic microwave background (CMB).

The small deviations from homogeneity and isotropy in the CMB are of uttermost importance since, most probably, they represent the ‘seeds’, which, via gravitational instability, have led to the formation of large-scale structure, galaxies and eventually solar systems with planets that support life in the Universe.

¹ If ‘analyticity’ is not assumed, the matter distribution could also be fractal and still statistically isotropic around each point. For a detailed elaboration of this idea and its comparison with observations see [Sylos Labini et al. \(1998\)](#).

Furthermore, we suppose that the initial fluctuations needed to trigger the process of gravitational instability stem from tiny quantum fluctuations that have been amplified during a period of inflationary expansion of the Universe. I consider this connection of the microscopic quantum world with the largest scales of the Universe to be of breathtaking philosophical beauty.

In this chapter we investigate the background Universe. We shall first discuss the geometry of a homogeneous and isotropic spacetime. Then we investigate two important events in the thermal history of the Universe. Finally, we study the paradigm of inflation. This chapter lays the basis for the following ones where we shall investigate *fluctuations* on the background, most of which can be treated in first-order perturbation theory.

1.2 The background geometry of the Universe

1.2.1 The Friedmann equations

In this section we assume a basic knowledge of general relativity. The notation and sign convention for the curvature tensor that we adopt are specified in Appendix A2.1.

Our Universe is described by a four-dimensional spacetime (\mathcal{M}, g) given by a pseudo-Riemannian manifold \mathcal{M} with metric g . A homogeneous and isotropic spacetime is one that admits a slicing into homogeneous and isotropic, i.e., maximally symmetric, 3-spaces. There is a preferred geodesic time coordinate τ , called ‘cosmic time’ such that the 3-spaces of constant time, $\Sigma_\tau = \{\mathbf{x} | (\tau, \mathbf{x}) \in \mathcal{M}\}$ are maximally symmetric spaces, hence spaces of constant curvature. The metric g is therefore of the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)\gamma_{ij} dx^i dx^j . \quad (1.6)$$

The function $a(\tau)$ is called the scale factor and γ_{ij} is the metric of a 3-space of constant curvature K . Depending on the sign of K this space is locally isometric to a 3-sphere ($K > 0$), a three-dimensional pseudo-sphere ($K < 0$) or flat, Euclidean space ($K = 0$). In later chapters of this book we shall mainly use ‘conformal time’ t defined by $a dt = d\tau$, so that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(t) (-dt^2 + \gamma_{ij} dx^i dx^j) . \quad (1.7)$$

The geometry and physics of homogeneous and isotropic solutions to Einstein’s equations was first investigated mathematically in the early twenties by Friedmann (1922) and physically as a description of the observed expanding Universe in 1927

by Lemaître.² Later, Robertson (1936), Walker (1936) and others rediscovered the Friedmann metric and studied several additional aspects. However, since we consider the contributions by Friedmann and Lemaître to be far more fundamental than the subsequent work, we shall call a homogeneous and isotropic solution to Einstein's equations a 'Friedmann–Lemaître universe' (FL universe) in this book.

It is interesting to note that the Friedmann solution breaks Lorentz invariance. Friedmann universes are not invariant under boosts, there is a preferred cosmic time τ , the proper time of an observer who sees a spatially homogeneous and isotropic universe. Like so often in physics, the Lagrangian and therefore also the field equations of general relativity are invariant under Lorentz transformations, but a specific solution in general is not. In that sense we are back to Newton's vision of an absolute time. But on small scales, e.g. the scale of a laboratory, this violation of Lorentz symmetry is, of course, negligible.

The topology is not determined by the metric and hence by Einstein's equations. There are many compact spaces of negative or vanishing curvature (e.g. the torus), but there are no infinite spaces with positive curvature. A beautiful treatment of the fascinating, but difficult, subject of the topology of spaces with constant curvature and their classification is given in [Wolf \(1974\)](#). Its applications to cosmology are found in [Lachieze-Rey & Luminet \(1995\)](#).

Forms of the metric γ , which we shall often use are

$$\gamma_{ij} dx^i dx^j = \frac{\delta_{ij} dx^i dx^j}{(1 + \frac{1}{4} K \rho^2)^2}, \quad (1.8)$$

$$\gamma_{ij} dx^i dx^j = dr^2 + \chi^2(r) (d\theta^2 + \sin^2(\theta) d\varphi^2), \quad (1.9)$$

$$\gamma_{ij} dx^i dx^j = \frac{dR^2}{1 - KR^2} + R^2 (d\theta^2 + \sin^2(\theta) d\varphi^2), \quad (1.10)$$

where in Eq. (1.8)

$$\rho^2 = \sum_{i,j=1}^3 \delta_{ij} x^i x^j, \quad \text{and} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases} \quad (1.11)$$

and in Eq. (1.9)

$$\chi(r) = \begin{cases} r & \text{in the Euclidean case, } K = 0, \\ \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) & \text{in the spherical case, } K > 0, \\ \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|}r) & \text{in the hyperbolic case, } K < 0. \end{cases} \quad (1.12)$$

Often one normalizes the scale factor such that $K = \pm 1$ whenever $K \neq 0$. One has, however, to keep in mind that in this case r and K become dimensionless and the

² In the English translation of ([Lemaître, 1927](#)) from 1931 Lemaître's somewhat premature but pioneering arguments that the observed Universe is actually expanding have been omitted.

scale factor a has the dimension of length. If $K = 0$ we can normalize a arbitrarily. We shall usually normalize the scale factor such that $a_0 = 1$ and the curvature is not dimensionless. The coordinate transformations which relate these coordinates are determined in Ex. 1.1.

Due to the symmetry of spacetime, the energy–momentum tensor can only be of the form

$$(T_{\mu\nu}) = \begin{pmatrix} -\rho g_{00} & \mathbf{0} \\ \mathbf{0} & P g_{ij} \end{pmatrix}. \quad (1.13)$$

There is no additional assumption going into this ansatz, such as the matter content of the Universe being an ideal fluid. It is a simple consequence of homogeneity and isotropy and is also verified for scalar field matter, a viscous fluid or free-streaming particles in a FL universe. As usual, the energy density ρ and the pressure P are defined as the time- and space-like eigenvalues of (T_{ν}^{μ}) .

The Einstein tensor can be calculated from the definition (A2.12) and Eqs. (A2.31)–(A2.38),

$$G_{00} = 3 \left[\left(\frac{a'}{a} \right)^2 + \frac{K}{a^2} \right] \quad (\text{cosmic time}), \quad (1.14)$$

$$G_{ij} = - \left(2a''a + a'^2 + K \right) \gamma_{ij} \quad (\text{cosmic time}), \quad (1.15)$$

$$G_{00} = 3 \left[\left(\frac{\dot{a}}{a} \right)^2 + K \right] \quad (\text{conformal time}), \quad (1.16)$$

$$G_{ij} = - \left(2 \left(\frac{\dot{a}}{a} \right)^{\bullet} + \left(\frac{\dot{a}}{a} \right)^2 + K \right) \gamma_{ij} \quad (\text{conformal time}). \quad (1.17)$$

The Einstein equations relate the Einstein tensor to the energy–momentum content of the Universe via $G_{\mu\nu} = 8\pi G T_{\mu\nu} - g_{\mu\nu} \Lambda$. Here Λ is the so-called cosmological constant. In a FL universe the Einstein equations become

$$\left(\frac{a'}{a} \right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} \quad (\text{cosmic time}), \quad (1.18)$$

$$2 \frac{a''}{a} + \frac{(a')^2}{a^2} + \frac{K}{a^2} = -8\pi G P + \Lambda \quad (\text{cosmic time}), \quad (1.19)$$

$$\left(\frac{\dot{a}}{a} \right)^2 + K = \frac{8\pi G}{3} a^2 \rho + \frac{a^2 \Lambda}{3} \quad (\text{conformal time}), \quad (1.20)$$

$$2 \left(\frac{\dot{a}}{a} \right)^{\bullet} + \left(\frac{\dot{a}}{a} \right)^2 + K = -8\pi G a^2 P + a^2 \Lambda \quad (\text{conformal time}). \quad (1.21)$$

Energy ‘conservation’, $T_{;\mu}^{\mu\nu} = 0$ yields

$$\dot{\rho} = -3(\rho + P) \left(\frac{\dot{a}}{a} \right) \quad \text{or, equivalently} \quad \rho' = -3(\rho + P) \left(\frac{a'}{a} \right). \quad (1.22)$$

This equation can also be obtained by differentiating Eq. (1.18) or (1.20) and inserting (1.19) or (1.21); it is a consequence of the contracted Bianchi identities (see Appendix A2.1). Eqs. (1.18)–(1.21) are the Friedmann equations. The quantity

$$H(\tau) \equiv \frac{a'}{a} = \frac{\dot{a}}{a^2} \equiv \mathcal{H}a^{-1}, \quad (1.23)$$

is called the Hubble rate or the Hubble parameter, where \mathcal{H} is the comoving Hubble parameter. At present, the Universe is expanding, so that $H_0 > 0$. We parametrize it by

$$H_0 = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1} \simeq 3.241 \times 10^{-18} h \text{ s}^{-1} \simeq 1.081 \times 10^{-28} h \text{ cm}^{-1}.$$

Observations show (Freedman *et al.*, 2001) that $h \simeq 0.72 \pm 0.1$. Eq. (1.22) is easily solved in the case $w = P/\rho = \text{constant}$. Then one finds

$$\rho = \rho_0 (a_0/a)^{3(1+w)}, \quad (1.24)$$

where ρ_0 and a_0 denote the value of the energy density and the scale factor at present time, τ_0 . In this book cosmological quantities indexed by a ‘0’ are evaluated today, $X_0 = X(\tau_0)$. For non-relativistic matter, $P_m = 0$, we therefore have $\rho_m \propto a^{-3}$ while for radiation (or any kind of massless particles) $P_r = \rho_r/3$ and hence $\rho_r \propto a^{-4}$. A cosmological constant corresponds to $P_\Lambda = -\rho_\Lambda$ and we obtain, as expected $\rho_\Lambda = \text{constant}$. If the curvature K can be neglected and the energy density is dominated by one component with $w = \text{constant}$, inserting Eq. (1.24) into the Friedmann equations yields the solutions

$$a \propto \tau^{2/3(1+w)} \propto t^{2/(1+3w)} \quad w = \text{constant} \neq -1, \quad (1.25)$$

$$a \propto \tau^{2/3} \propto t^2 \quad w = 0, \quad (\text{dust}), \quad (1.26)$$

$$a \propto \tau^{1/2} \propto t \quad w = 1/3, \quad (\text{radiation}), \quad (1.27)$$

$$a \propto \exp(H\tau) \propto 1/|t| \quad w = -1, \quad (\text{cosmol. const.}). \quad (1.28)$$

It is interesting to note that if $w < -1$, so-called ‘phantom matter’, we have to choose $\tau < 0$ to obtain an expanding universe and the scale factor diverges in finite time, at $\tau = 0$. This is the so-called ‘big rip’. Phantom matter has many problems but it is discussed in connection with the supernova type 1a (SN1a) data, which are compatible with an equation of state with $w < -1$ or with an ordinary cosmological constant (Caldwell *et al.*, 2003). For $w < -\frac{1}{3}$ the time coordinate t has to be chosen as negative for the Universe to expand and spacetime cannot be

continued beyond $t = 0$. But $t = 0$ corresponds to a cosmic time, the proper time of a static observer, $\tau = \infty$; this is not a singularity. (The geodesics can be continued until affine parameter ∞ .)

We also introduce the adiabatic sound speed c_s determined by

$$c_s^2 = \frac{P'}{\rho'} = \frac{\dot{P}}{\dot{\rho}}. \quad (1.29)$$

From this definition and Eq. (1.22) it is easy to see that

$$\dot{w} = 3\mathcal{H}(1+w)(w - c_s^2). \quad (1.30)$$

Hence $w = \text{constant}$ if and only if $w = c_s^2$ or $w = -1$. Note that already in a simple mixture of matter and radiation $w \neq c_s^2 \neq \text{constant}$ (see Ex. 1.3).

Eq. (1.18) implies that for a critical value of the energy density given by

$$\rho(\tau) = \rho_c(\tau) = \frac{3H^2}{8\pi G} \quad (1.31)$$

the curvature and the cosmological constant vanish. The value ρ_c is called the critical density. The ratio $\Omega_X = \rho_X/\rho_c$ is the ‘density parameter’ of the component X . It indicates the fraction that the component X contributes to the expansion of the Universe. We shall make use especially of

$$\Omega_r \equiv \Omega_r(\tau_0) = \frac{\rho_r(\tau_0)}{\rho_c(\tau_0)}, \quad (1.32)$$

$$\Omega_m \equiv \Omega_m(\tau_0) = \frac{\rho_m(\tau_0)}{\rho_c(\tau_0)}, \quad (1.33)$$

$$\Omega_K \equiv \Omega_K(\tau_0) = \frac{-K}{a_0^2 H_0^2}, \quad (1.34)$$

$$\Omega_\Lambda \equiv \Omega_\Lambda(\tau_0) = \frac{\Lambda}{3H_0^2}. \quad (1.35)$$

1.2.2 The ‘big bang’ and ‘big crunch’ singularities

We can absorb the cosmological constant into the energy density and pressure by redefining

$$\rho_{\text{eff}} = \rho + \frac{\Lambda}{8\pi G}, \quad P_{\text{eff}} = P - \frac{\Lambda}{8\pi G}.$$

Since Λ is a constant and $\rho_{\text{eff}} + P_{\text{eff}} = \rho + P$, the conservation equation (1.22) still holds. A first interesting consequence of the Friedmann equations is obtained

when subtracting Eq. (1.18) from (1.19). This yields

$$\frac{a''}{a} = -\frac{4\pi G}{3}(\rho_{\text{eff}} + 3P_{\text{eff}}). \quad (1.36)$$

Hence if $\rho_{\text{eff}} + 3P_{\text{eff}} > 0$, the Universe is decelerating. Furthermore, Eqs. (1.22) and (1.36) then imply that in an expanding and decelerating universe

$$\frac{\rho'_{\text{eff}}}{\rho_{\text{eff}}} < -2\frac{a'}{a},$$

so that ρ decays faster than $1/a^2$. If the curvature is positive, $K > 0$, this implies that at some time in the future, τ_{max} , the density has dropped down to the value of the curvature term, $K/a^2(\tau_{\text{max}}) = 8\pi G\rho_{\text{eff}}(\tau_{\text{max}})$. Then the Universe stops expanding and recollapses. Furthermore, this is independent of curvature, as a' decreases the curve $a(\tau)$ is concave and thus cuts the $a = 0$ line at some finite time in the past. This moment of time is called the ‘big bang’. The spatial metric vanishes at this value of τ , which we usually choose to be $\tau = 0$; and spacetime cannot be continued to earlier times. This is not a coordinate singularity. From the Ricci tensor given in Eqs. (A2.31) and (A2.32) one obtains the Riemann scalar

$$R = 6 \left[\frac{a''}{a} + \left(\frac{a'}{a} \right)^2 + \frac{K}{a^2} \right],$$

which also diverges if $a \rightarrow 0$. Also the energy density, which grows faster than $1/a^2$ as $a \rightarrow 0$ diverges at the big bang.

If the curvature K is positive, the Universe contracts after $\tau = \tau_{\text{max}}$ and, since the graph $a(\tau)$ is convex, reaches $a = 0$ at some finite time τ_c , the time of the ‘big crunch’. The big crunch is also a physical singularity beyond which spacetime cannot be continued.

It is important to note that this behaviour of the scale factor can only be implied if the so-called ‘strong energy condition’ holds, $\rho_{\text{eff}} + 3P_{\text{eff}} > 0$. This is illustrated in Fig. 1.1.

1.2.3 Cosmological distance measures

It is notoriously difficult to measure distances in the Universe. The position of an object in the sky gives us its angular coordinates, but how far away is the object from us? This problem has plagued cosmology for centuries. It was only Hubble, who discovered around 1915–1920 that the ‘spiral nebulae’ are actually not situated inside our own galaxy but much further away. This then led to the discovery of the expansion of the Universe.

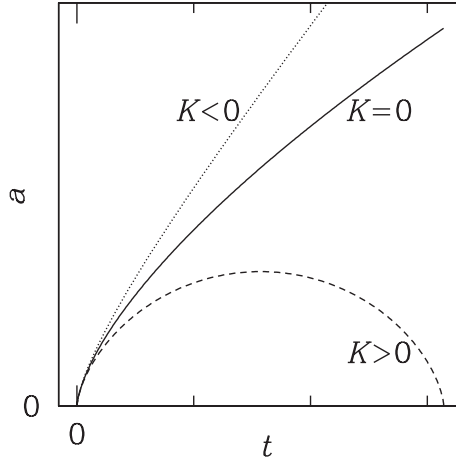


Fig. 1.1. The kinematics of the scale factor in a Friedmann–Lemaître universe which satisfies the strong energy condition, $\rho_{\text{eff}} + 3P_{\text{eff}} > 0$.

For cosmologically distant objects, a third coordinate, which is nowadays relatively easy to obtain, is the redshift z experienced by the photons emitted from the object. A given spectral line with intrinsic wavelength λ is redshifted due to the expansion of the Universe. If it is emitted at some time τ , it reaches us today with wavelength $\lambda_0 = \lambda a_0/a(\tau) = (1+z)\lambda$. This leads to the definition of the cosmic redshift

$$z(\tau) + 1 = \frac{a_0}{a(\tau)}. \quad (1.37)$$

On the other hand, an object at physical distance $d = a_0 r$ away from us, at redshift $z \ll 1$, recedes with speed $v = H_0 d$. To the lowest order in z , we have $\tau_0 - \tau \approx d$ and $a_0 \approx a(\tau) + a'(\tau_0 - \tau)$, so that

$$1 + z \approx 1 + \frac{a'}{a}(\tau_0 - \tau) \approx 1 + H_0 d.$$

For objects that are sufficiently close, $z \ll 1$ we therefore have $v \approx z$ and hence $H_0 = v/d$. This is the method usually applied to measure the Hubble constant.

There are different ways to measure distances in cosmology all of which give the same result in a Minkowski universe but differ in an expanding universe. They are, however, simply related as we shall see.

One possibility is to define the distance D_A to a certain object of given physical size Δ seen at redshift z_1 such that the angle subtended by the object is given by

$$\vartheta = \Delta/D_A, \quad D_A = \Delta/\vartheta. \quad (1.38)$$

This is the angular diameter distance, see Fig. 1.2.



Fig. 1.2. The two ends of the object emit a flash simultaneously from A and B at z_1 which reaches us today. The angular diameter distance to A (or B) is defined by $D_A = \Delta/\vartheta$.

We now derive the expression

$$D_A(z) = \frac{1}{\sqrt{|\Omega_K|}H_0(1+z)}\chi\left(\sqrt{|\Omega_K|}H_0\int_0^z\frac{dz'}{H(z')}\right), \quad (1.39)$$

for the angular diameter distance to redshift z . In a given cosmological model, this allows us to express the angular diameter distance for a given redshift as a function of the cosmological parameters.

To derive Eq. (1.39) we use the coordinates introduced in Eq. (1.9). Without loss of generality we set $r = 0$ at our position. We consider an object of physical size Δ at redshift z_1 simultaneously emitting a flash at both of its ends A and B . Hence $r = r_1 = t_0 - t_1$ at the position of the flashes, A and B at redshift z_1 . If Δ denotes the physical arc length between A and B we have $\Delta = a(t_1)\chi(r_1)\vartheta = a(t_1)\chi(t_0 - t_1)\vartheta$, i.e.,

$$\vartheta = \frac{\Delta}{a(t_1)\chi(t_0 - t_1)}. \quad (1.40)$$

According to Eq. (1.38) the angular diameter distance to t_1 or z_1 is therefore given by

$$a(t_1)\chi(t_0 - t_1) \equiv D_A(z_1). \quad (1.41)$$

To obtain an expression for $D_A(z)$ in terms of the cosmic density parameters and the redshift, we have to calculate $(t_0 - t_1)(z_1)$.

Note that in the case $K = 0$ we can normalize the scale factor a as we want, and it is convenient to choose $a_0 = 1$, so that comoving scales become physical scales today. However, for $K \neq 0$, we have already normalized a such that $K = \pm 1$ and $\chi(r) = \sin r$ or $\sinh r$. In this case, we have no normalization constant left and a_0 has the dimension of a length. The present spatial curvature of the Universe then is $\pm 1/a_0^2$.

The Friedmann equation Eq. (1.20) reads

$$\dot{a}^2 = \frac{8\pi G}{3}a^4\rho + \frac{1}{3}\Lambda a^4 - Ka^2, \quad (1.42)$$

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