

# PURITY, SPECTRA AND LOCALISATION

Mike Prest

CAMBRIDGE

CAMBRIDGE

more information – [www.cambridge.org/9780521873086](http://www.cambridge.org/9780521873086)



---

## **Purity, Spectra and Localisation**

This book is an account of a fruitful interaction between algebra, mathematical logic, and category theory. It is possible to associate a topological space to the category of modules over any ring. This space, the Ziegler spectrum, is based on the indecomposable pure-injective modules. Although the Ziegler spectrum arose within the model theory of modules and plays a central role in that subject, this book concentrates on its algebraic aspects and uses.

The central aim is to understand modules and the categories they form through associated structures and dimensions which reflect the complexity of these, and similar, categories. The structures and dimensions considered arise through the application of ideas and methods from model theory and functor category theory. Purity and associated notions are central, localisation is an ever-present theme and various types of spectrum play organising roles.

This book presents a unified account of material which is often presented from very different viewpoints and it clarifies the relationships between these various approaches. It may be used as an introductory graduate-level text, since it provides relevant background material and a wealth of illustrative examples. An extensive index and thorough referencing also make this book an ideal, comprehensive reference.

MIKE PREST is Professor of Pure Mathematics at the University of Manchester.

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

<http://www.cambridge.org/uk/series/sSeries.asp?code=EOM>

- 69 T. Beth, D. Jungnickel, and H. Lenz *Design Theory I, 2nd edn*
- 70 A. Pietsch and J. Wenzel *Orthonormal Systems for Banach Space Geometry*
- 71 G. E. Andrews, R. Askey and R. Roy *Special Functions*
- 72 R. Ticciati *Quantum Field Theory for Mathematicians*
- 73 M. Stern *Semimodular Lattices*
- 74 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations I*
- 75 I. Lasiecka and R. Triggiani *Control Theory for Partial Differential Equations II*
- 76 A. A. Ivanov *Geometry of Sporadic Groups I*
- 77 A. Schinzel *Polynomials with Special Regard to Reducibility*
- 78 H. Lenz, T. Beth, and D. Jungnickel *Design Theory II, 2nd edn*
- 79 T. Palmer *Banach Algebras and the General Theory of \*-Algebras II*
- 80 O. Stormark *Lie's Structural Approach to PDE Systems*
- 81 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables*
- 82 J. P. Mayberry *The Foundations of Mathematics in the Theory of Sets*
- 83 C. Foias, O. Manley, R. Rosa and R. Temam *Navier–Stokes Equations and Turbulence*
- 84 B. Polster and G. Steinke *Geometries on Surfaces*
- 85 R. B. Paris and D. Kaminski *Asymptotics and Mellin–Barnes Integrals*
- 86 R. McEliece *The Theory of Information and Coding, 2nd edn*
- 87 B. Magurn *Algebraic Introduction to K-Theory*
- 88 T. Mora *Solving Polynomial Equation Systems I*
- 89 K. Bichteler *Stochastic Integration with Jumps*
- 90 M. Lothaire *Algebraic Combinatorics on Words*
- 91 A. A. Ivanov and S. V. Shpectorov *Geometry of Sporadic Groups II*
- 92 P. McMullen and E. Schulte *Abstract Regular Polytopes*
- 93 G. Gierz et al. *Continuous Lattices and Domains*
- 94 S. Finch *Mathematical Constants*
- 95 Y. Jabri *The Mountain Pass Theorem*
- 96 G. Gasper and M. Rahman *Basic Hypergeometric Series, 2nd edn*
- 97 M. C. Pedicchio and W. Tholen (eds.) *Categorical Foundations*
- 98 M. E. H. Ismail *Classical and Quantum Orthogonal Polynomials in One Variable*
- 99 T. Mora *Solving Polynomial Equation Systems II*
- 100 E. Olivieri and M. Eulália Vares *Large Deviations and Metastability*
- 101 A. Kushner, V. Lychagin and V. Rubtsov *Contact Geometry and Nonlinear Differential Equations*
- 102 L. W. Beineke and R. J. Wilson (eds.) with P. J. Cameron *Topics in Algebraic Graph Theory*
- 103 O. Staffans *Well-Posed Linear Systems*
- 104 J. M. Lewis, S. Lakshmivarahan and S. Dhall *Dynamic Data Assimilation*
- 105 M. Lothaire *Applied Combinatorics on Words*
- 106 A. Markoe *Analytic Tomography*
- 107 P. A. Martin *Multiple Scattering*
- 108 R. A. Brualdi *Combinatorial Matrix Classes*
- 110 M.-J. Lai and L. L. Schumaker *Spline Functions on Triangulations*
- 111 R. T. Curtis *Symmetric Generation of Groups*
- 112 H. Salzmann, T. Grundhöfer, H. Hähl and R. Löwen *The Classical Fields*
- 113 S. Peszat and J. Zabczyk *Stochastic Partial Differential Equations with Lévy Noise*
- 114 J. Beck *Combinatorial Games*
- 115 L. Barreira and Y. Pesin *Nonuniform Hyperbolicity*
- 116 D. Z. Arov and H. Dym *J-Contractive Matrix Valued Functions and Related Topics*
- 117 R. Glowinski, J.-L. Lions and J. He *Exact and Approximate Controllability for Distributed Parameter Systems*
- 118 A. A. Borovkov and K. A. Borovkov *Asymptotic Analysis of Random Walks*
- 119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
- 120 T. Nishiura *Absolute Measurable Spaces*
- 121 M. Prest *Purity, Spectra and Localisation*
- 122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions: From Euler's Point of View*
- 123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivities*
- 124 F. W. King *Hilbert Transforms I*
- 125 F. W. King *Hilbert Transforms II*
- 126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*

---

# Purity, Spectra and Localisation

MIKE PREST  
*University of Manchester*



---

CAMBRIDGE UNIVERSITY PRESS  
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press  
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
Information on this title: [www.cambridge.org/9780521873086](http://www.cambridge.org/9780521873086)

© M. Prest 2009

This publication is in copyright. Subject to statutory exception  
and to the provisions of relevant collective licensing agreements,  
no reproduction of any part may take place without  
the written permission of Cambridge University Press.

First published 2009

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

ISBN 978-0-521-87308-6 hardback

---

Cambridge University Press has no responsibility for the persistence or  
accuracy of URLs for external or third-party internet websites referred to  
in this publication, and does not guarantee that any content on such  
websites is, or will remain, accurate or appropriate.

---

---

*To Abigail, Stephen and Iain*





---

# Contents

---

<i>Preface</i>	xv
<i>Introduction</i>	xvii
Conventions and notations	xxiii
Selected notations list	xxviii
<b>Part I Modules</b>	<b>1</b>
<b>1 Pp conditions</b>	<b>3</b>
1.1 Pp conditions	3
1.1.1 Pp-definable subgroups of modules	3
1.1.2 The functor corresponding to a pp condition	10
1.1.3 The lattice of pp conditions	15
1.1.4 Further examples	17
1.2 Pp conditions and finitely presented modules	18
1.2.1 Pp-types	18
1.2.2 Finitely presented modules and free realisations	20
1.2.3 Irreducible pp conditions	29
1.3 Elementary duality of pp conditions	30
1.3.1 Elementary duality	30
1.3.2 Elementary duality and tensor product	33
1.3.3 Character modules and duality	35
1.3.4 Pp conditions in tensor products	38
1.3.5 Mittag–Leffler modules	39
<b>2 Purity</b>	<b>43</b>
2.1 Purity	43
2.1.1 Pure-exact sequences	43
2.1.2 Pure-projective modules	51
2.1.3 Purity and tensor product	53

2.2	Pure global dimension	54
2.3	Absolutely pure and flat modules	57
2.3.1	Absolutely pure modules	57
2.3.2	Flat modules	59
2.3.3	Coherent modules and coherent rings	63
2.3.4	Von Neumann regular rings	66
2.4	Purity and structure of finitely presented modules	68
2.4.1	Direct sum decomposition of finitely presented modules	68
2.4.2	Purity over Dedekind domains and RD rings	73
2.4.3	Pp conditions over serial rings	77
2.5	Pure-projective modules over uniserial rings	80
<b>3</b>	<b>Pp-pairs and definable subcategories</b>	<b>85</b>
3.1	Pp conditions and morphisms between pointed modules	85
3.2	Pp-pairs	87
3.2.1	Lattices of pp conditions	87
3.2.2	The category of pp-pairs	91
3.3	Reduced products and pp-types	99
3.3.1	Reduced products	99
3.3.2	Pp conditions in reduced products	102
3.3.3	Realising pp-types in reduced products	104
3.4	Definable subcategories	104
3.4.1	Definable subcategories	104
3.4.2	Duality and definable subcategories	112
3.4.3	Further examples of definable subcategories	114
3.4.4	Covariantly finite subcategories	118
<b>4</b>	<b>Pp-types and pure-injectivity</b>	<b>124</b>
4.1	Pp-types with parameters	124
4.2	Algebraic compactness	127
4.2.1	Algebraically compact modules	127
4.2.2	Linear compactness	131
4.2.3	Topological compactness	133
4.2.4	Algebraically compact $\mathbb{Z}$ -modules	135
4.2.5	Algebraic compactness of ultraproducts	136
4.3	Pure-injectivity	137
4.3.1	Pure-injective modules	138
4.3.2	Algebraically compact = pure-injective	143
4.3.3	Pure-injective hulls	144
4.3.4	Pure-injective extensions via duality	151
4.3.5	Hulls of pp-types	153
4.3.6	Indecomposable pure-injectives and irreducible pp-types	157

4.3.7	Pure-injective hulls of finitely presented modules	162
4.3.8	Krull–Schmidt rings	164
4.3.9	Linking and quotients of pp-types	167
4.3.10	Irreducible pp-types and indecomposable direct summands	169
4.4	Structure of pure-injective modules	171
4.4.1	Decomposition of pure-injective modules	171
4.4.2	$\Sigma$ -pure-injective modules	173
4.4.3	Modules of finite endolength	179
4.4.4	Characters	183
4.4.5	The ascending chain condition on pp-definable subgroups	187
4.5	Representation type and pure-injective modules	188
4.5.1	Pure-semisimple rings	189
4.5.2	Finite length modules over pure-semisimple rings	192
4.5.3	Rings of finite representation type	195
4.5.4	The pure-semisimplicity conjecture	198
4.5.5	Generic modules and representation type	200
4.6	Cotorsion, flat and pure-injective modules	205
<b>5</b>	<b>The Ziegler spectrum</b>	<b>209</b>
5.1	The Ziegler spectrum	209
5.1.1	The Ziegler spectrum via definable subcategories	210
5.1.2	Ziegler spectra via pp-pairs: proofs	217
5.1.3	Ziegler spectra via morphisms	220
5.2	Examples	221
5.2.1	The Ziegler spectrum of a Dedekind domain	221
5.2.2	Spectra over RD rings	227
5.2.3	Other remarks	228
5.3	Isolation, density and Cantor–Bendixson rank	229
5.3.1	Isolated points and minimal pairs	230
5.3.2	The isolation condition	239
5.3.3	Minimal pairs and left almost split maps	248
5.3.4	Density of (hulls of) finitely presented modules	250
5.3.5	Neg-isolated points and elementary cogenerators	253
5.3.6	Cantor–Bendixson analysis of the spectrum	260
5.3.7	The full support topology	263
5.4	Duality of spectra	266
5.5	Maps between spectra	273
5.5.1	Epimorphisms of rings	274
5.5.2	Representation embeddings	276
5.6	The dual-Ziegler topology	278

<b>6</b>	<b>Rings of definable scalars</b>	280
6.1	Rings of definable scalars	280
6.1.1	Actions defined by pp conditions	280
6.1.2	Rings of definable scalars and epimorphisms	284
6.1.3	Rings of definable scalars and localisation	288
6.1.4	Duality and rings of definable scalars	291
6.1.5	Rings of definable scalars over a PI Dedekind domain	292
6.1.6	Rings of type-definable scalars	292
<b>7</b>	<b>M-dimension and width</b>	298
7.1	Dimensions on lattices	298
7.2	M-dimension	302
7.2.1	Calculating m-dimension	304
7.2.2	Factorisable systems of morphisms and m-dimension	305
7.3	Width	311
7.3.1	Width and superdecomposable pure-injectives	311
7.3.2	Existence of superdecomposable pure-injectives	318
<b>8</b>	<b>Examples</b>	325
8.1	Spectra of artin algebras	325
8.1.1	Points of the spectrum	325
8.1.2	Spectra of tame hereditary algebras	327
8.1.3	Spectra of some string algebras	343
8.1.4	Spectra of canonical algebras	350
8.2	Further examples	352
8.2.1	Ore and RD domains	352
8.2.2	Spectra over HNP rings	355
8.2.3	Pseudofinite representations of $sl_2$	358
8.2.4	Verma modules over $sl_2$	363
8.2.5	The spectrum of the first Weyl algebra and related rings	366
8.2.6	Spectra of V-rings and differential polynomial rings	374
8.2.7	Spectra of serial rings	377
8.2.8	Spectra of uniserial rings	382
8.2.9	Spectra of pullback rings	387
8.2.10	Spectra of von Neumann regular rings	390
8.2.11	Commutative von Neumann regular rings	393
8.2.12	Indiscrete rings and almost regular rings	395
<b>9</b>	<b>Ideals in mod-<math>R</math></b>	399
9.1	The radical of mod- $R$	399
9.1.1	Ideals in mod- $R$	399
9.1.2	The transfinite radical of mod- $R$	401

9.1.3	Powers of the radical and factorisation of morphisms	403
9.1.4	The transfinite radical and Krull–Gabriel/ $m$ -dimension	404
9.2	Fp-idempotent ideals	408
<i>Appendix A Model theory</i>		411
A.1	Model theory of modules	411
<b>Part II Functors</b>		417
<b>10</b>	<b>Finitely presented functors</b>	419
10.1	Functor categories	419
10.1.1	Functors and modules	419
10.1.2	The Yoneda embedding	425
10.1.3	Representable functors and projective objects in functor categories	427
10.2	Finitely presented functors in $(\text{mod-}R, \mathbf{Ab})$	429
10.2.1	Local coherence of $(\text{mod-}R, \mathbf{Ab})$	430
10.2.2	Projective dimension of finitely presented functors	435
10.2.3	Minimal free realisations and local functors	439
10.2.4	Pp conditions over rings with many objects	442
10.2.5	Finitely presented functors = pp-pairs	443
10.2.6	Examples of finitely presented functors	446
10.2.7	Free abelian categories	448
10.2.8	Extending functors along direct limits	450
10.3	Duality of finitely presented functors	452
10.4	Finitistic global dimension	456
<b>11</b>	<b>Serre subcategories and localisation</b>	459
11.1	Localisation in Grothendieck categories	459
11.1.1	Localisation	459
11.1.2	Finite-type localisation in locally finitely generated categories	468
11.1.3	Elementary localisation and locally finitely presented categories	473
11.1.4	Finite-type localisation in locally coherent categories	480
11.1.5	Pp conditions in locally finitely presented categories	485
11.2	Serre subcategories and ideals	487
11.2.1	Annihilators of ideals of $\text{mod-}R$	487
11.2.2	Duality of Serre subcategories	491
<b>12</b>	<b>The Ziegler spectrum and injective functors</b>	492
12.1	Making modules functors	492
12.1.1	The tensor embedding	492

12.1.2	Injectives in the category of finitely presented functors	500
12.1.3	The Ziegler spectrum revisited yet again	502
12.2	Pp-types, subfunctors of $({}_R R^n, -)$ , finitely generated functors	503
12.3	Definable subcategories again	508
12.4	Ziegler spectra and Serre subcategories: summary	515
12.5	Hulls of simple functors	518
12.6	A construction of pp-types without width	521
12.7	The full support topology again	525
12.8	Rings of definable scalars again	527
<b>13</b>	<b>Dimensions</b>	<b>533</b>
13.1	Dimensions	533
13.1.1	Dimensions via iterated localisation	533
13.1.2	Dimensions on lattices of finitely presented subfunctors	535
13.2	Krull–Gabriel dimension	538
13.2.1	Definition and examples	538
13.2.2	Gabriel dimension and Krull–Gabriel dimension	542
13.3	Locally simple objects	544
13.4	Uniserial dimension	545
<b>14</b>	<b>The Zariski spectrum and the sheaf of definable scalars</b>	<b>549</b>
14.1	The Gabriel–Zariski spectrum	549
14.1.1	The Zariski spectrum through representations	550
14.1.2	The Gabriel–Zariski and rep-Zariski spectra	552
14.1.3	Rep-Zariski = dual-Ziegler	557
14.1.4	The sheaf of locally definable scalars	560
14.2	Topological properties of $\text{Zar}_R$	562
14.3	Examples	564
14.3.1	The rep-Zariski spectrum of a PI Dedekind domain	564
14.3.2	The sheaf of locally definable scalars of a PI Dedekind domain	566
14.3.3	The presheaf of definable scalars of a PI HNP ring	569
14.3.4	The presheaf of definable scalars of a tame hereditary artin algebra	572
14.3.5	Other examples	574
14.4	The spectrum of a commutative coherent ring	575
<b>15</b>	<b>Artin algebras</b>	<b>582</b>
15.1	Quivers and representations	582
15.1.1	Representations of quivers	582
15.1.2	The Auslander–Reiten quiver of an artin algebra	586
15.1.3	Tubes and generalised tubes	588
15.2	Duality over artin algebras	593

15.3	Ideals in $\text{mod-}R$ when $R$ is an artin algebra	596
15.4	$\text{mdim} \neq 1$ for artin algebras	599
15.5	Modules over group rings	600
<b>16</b>	<b>Finitely accessible and presentable additive categories</b>	<b>603</b>
16.1	Finitely accessible additive categories	603
16.1.1	Representation of finitely accessible additive categories	603
16.1.2	Purity in finitely accessible categories	609
16.1.3	Conjugate and dual categories	611
16.2	Categories generated by modules	612
16.3	Categories of presheaves and sheaves	617
16.3.1	Categories of presheaves	618
16.3.2	Finite-type localisation in categories of presheaves	619
16.3.3	The category $\text{Mod-}\mathcal{O}_X$ : local finite presentation	622
16.3.4	The category $\text{Mod-}\mathcal{O}_X$ : local finite generation	625
16.3.5	Pp conditions in categories of sheaves	627
<b>17</b>	<b>Spectra of triangulated categories</b>	<b>630</b>
17.1	Triangulated categories: examples	631
17.2	Compactly generated triangulated categories	636
17.2.1	Brown representability	637
17.2.2	The functor category of a compactly generated triangulated category	640
17.3	Purity in compactly generated triangulated categories	642
17.3.1	The Ziegler spectrum of a compactly generated triangulated category	648
17.3.2	The Ziegler spectrum of $\mathcal{D}(R)$	649
17.4	Localisation	651
17.5	The spectrum of the cohomology ring of a group ring	653
	<i>Appendix B Languages for definable categories</i>	656
B.1	Languages for finitely accessible categories	656
B.1.1	Languages for modules	659
B.2	Imaginaries	660
	<i>Appendix C A model theory/functor category dictionary</i>	663
	<b>Part III Definable categories</b>	<b>665</b>
<b>18</b>	<b>Definable categories and interpretation functors</b>	<b>667</b>
18.1	Definable categories	667
18.1.1	Definable subcategories	667
18.1.2	Exactly definable categories	671

---

18.1.3	Recovering the definable structure	674
18.1.4	Definable categories	675
18.2	Functors between definable categories	677
18.2.1	Interpretation functors	680
18.2.2	Examples of interpretations	685
18.2.3	Tilting functors	687
18.2.4	Another example: lattices over groups	690
18.2.5	Definable functors and Ziegler spectra	692
<i>Appendix D Model theory of modules: an update</i>		696
<i>Appendix E Some definitions</i>		703
<i>Main examples</i>		718
<i>Bibliography</i>		720
<i>Index</i>		754



---

## Preface

---

In his paper [726], on the model theory of modules, Ziegler associated a topological space to the category of modules over any ring. The points of this space are certain indecomposable modules and the definition of the topology was in terms of concepts from model theory. This space, now called the Ziegler spectrum, has played a central role in the model theory of modules. More than one might have expected, this space and the ideas surrounding it have turned out to be interesting and useful for purely algebraic reasons. This book is mostly about these algebraic aspects.

The central aim is a better understanding of the category of modules over a ring. Over most rings this category is far too complicated to describe completely so one must be content with aiming to classify the most significant types of modules and to understand more global aspects in just a broad sense, for example by finding some geometric or topological structure that organises some aspect of the category and which reflects the complexity of the category.

By “significant types of modules” one might mean the irreducible representations or the “finite” (finite-dimensional/finitely generated) ones. Here I mean the pure-injective modules. Over many rings this class of modules includes, directly or by proxy, the “finite” ones. There is a decomposition theorem which means that for most purposes we can concentrate on the indecomposable pure-injective modules.

The Ziegler spectrum is one example of an “organising” structure; it is a topological space whose points are the isomorphism classes of indecomposable pure-injectives, and the Cantor–Bendixson analysis of this space does reflect various aspects of complexity of the module category. There are associated structures: the category of functors on finitely presented modules; the lattice of pp conditions; the presheaf of rings of definable scalars. Various dimensions and ranks are defined on these and they are all linked together.

Here I present a cluster of concepts, techniques, results and applications. The inputs are from algebra, model theory and category theory and many of the results and methods are hybrids of these. The way in which these combine here is something which certainly I have found fascinating. The applications are mainly algebraic though not confined to modules since everything works in good enough abelian categories. Again, I have been pleasantly surprised by the extent to which what began in model theory has had applications and ramifications well beyond that subject.

Around 2000 it seemed to me that the central part of the subject had pretty well taken shape, though mainly in the minds of those who were working with it and using it. Much was not written down and there was no unified account so, foolhardily, I decided to write one. This book, which is the result, has far outgrown my original intentions (in length, time, effort, . . . ). In the category of books it is a pushout of a graduate-level course and a work of reference.

---

# Introduction

---

The Ziegler spectrum,  $Zg_R$ , of a ring  $R$  is a topological space. It is defined in terms of the category of  $R$ -modules and, although a Ziegler spectrum can be assigned to much more general categories, let us stay with rings and modules at the beginning. The points of  $Zg_R$  are certain modules, more precisely they are the isomorphism types of indecomposable pure-injective (also called algebraically compact) right  $R$ -modules. Any injective module is pure-injective but usually there are more, indeed a ring is von Neumann regular exactly if there are no other pure-injective modules (2.3.22). If  $R$  is an algebra over a field  $k$ , then any module which is finite-dimensional as a  $k$ -vector space is pure-injective (4.2.6). Every finite module is pure-injective (4.2.6). Another example is the ring of  $p$ -adic integers, regarded as a module over any ring between  $\mathbb{Z}$  and itself (4.2.8). The pure-injective modules mentioned so far are either “small” or, although large in some sense, have some kind of completeness property. There is something of a general point there but, as it stands, it is too vague: not all “small” modules are pure-injective. For example, the finite-length modules over the first Weyl algebra,  $A_1(k)$ , over a field  $k$  of characteristic zero are not pure-injective (8.2.35). They are small in the sense of being of finite length, but large in that they are infinite-dimensional. Nevertheless each indecomposable finite-length module over the first Weyl algebra has a pure-injective hull (a minimal pure, pure-injective extension, see Section 4.3.3) which is indecomposable. Indeed, associating to a finite-length module its pure-injective hull gives a bijection between the set of (isomorphism types of) indecomposable finite-length modules over  $A_1(k)$  and a subset of the Ziegler spectrum of  $A_1(k)$  (8.2.39).

Ziegler defined the topology of this space in terms of solution sets to certain types of linear conditions (5.1.21) but there are equivalent definitions: in terms of morphisms between finitely presented modules (5.1.25); also in terms of finitely presented functors (10.2.45). Ziegler showed that understanding this space, in the very best case obtaining a list of points and an explicit description of the topology,

is the key to answering most questions about the model theory of modules over the given ring. For that aspect one may consult [495]. Most subsequent advances have been driven more by algebraic than model-theoretic questions though much of what is here can be reformulated to say something about the model theory of modules.

Over some rings there is a complete description of the Ziegler spectrum (see especially Section 5.2 and Chapter 8); for  $R = \mathbb{Z}$  the list of points is due to Kaplansky [330], see Section 5.2.1.

A module which is of finite length over its endomorphism ring is pure-injective (4.2.6, 4.4.24) but, unless the ring is right pure-semisimple (conjecturally equivalent to being of finite representation type, see Section 4.5.4), one should expect there to be “large” points of the Ziegler spectrum. Over an artin algebra a precise expression of this is the existence of infinite-dimensional indecomposable pure-injectives (5.3.40) if the ring is not of finite representation type. This is an easy consequence of compactness of the Ziegler spectrum of a ring (5.1.23). Even if one is initially interested in “small” modules, for example, finite-dimensional representations, the “large” modules may appear quite naturally: often the latter parametrise, in some sense or another, natural families of the former (e.g. 5.2.2, Sections 4.5.5 and 15.1.3). Examples of such large parametrising modules are the generic modules of Crawley-Boevey (Section 4.5.5).

A natural context for most of the results here is that of certain, “definable”, subcategories of locally finitely presented abelian categories: the latter are, roughly, abelian categories in which objects are determined by their “elements”, see Chapter 16. There are reasons for working in the more general context beyond simply wider applicability. Auslander and coworkers, in particular Reiten, showed how, if one is interested in finite-dimensional representations of a finite-dimensional  $k$ -algebra, it is extremely useful to move to the, admittedly more abstract, category of  $k$ -linear functors from the category of these representations to the category of  $k$ -vector spaces. It turns out that, in describing ideas around the Ziegler spectrum, moving to an associated such functor category often clarifies concepts and simplifies arguments (and leads to new results!). By this route one may also dispense with the terminology of model theory, though model theory still provides concepts and techniques, and replace it with the more widely known terminology of categories and functors. In particular, one may define the Ziegler spectrum of a ring as a topology on the (set of isomorphism types of) indecomposable injectives in the corresponding functor category (12.1.17). This topology is dual (Section 5.6) to another topology, on the same set, which one might regard as the (Gabriel–)Zariski spectrum of the functor category (14.1.6).

This equivalence between model-theoretic and functorial methods is best explained by an equivalence, 10.2.30, between the model-theoretic category of

“imaginaries” (in the sense of Section B.2) and the category of finitely presented functors. There is some discussion of this equivalence between methods, and see Appendix C, but I have tended to avoid the terminology of model theory, except where I consider it to be particularly efficient or where there is no algebraic equivalent. That is simply because it is less well known, though I hope that one effect of this book will be that some people become a little more comfortable with it. This does mean that those already familiar with model theory might have to work a bit harder than they expected: the terminological adjustments are quite slight, but the conceptual adjustments (the use of functorial methods) may well require more effort. It should be noted that much of the relevant literature does assume familiarity with the most basic ideas from model theory.

As mentioned already, it is possible to give definitions (5.1.1, 5.1.25) of the Ziegler spectrum of a ring purely in terms of its category of modules, that is, without reference to model theory or to any “external” (functor) category. In fact the book begins by taking a “naïve”, element-wise, view of modules and gradually, though not monotonically, takes an increasingly “sophisticated” view. I discuss this now.

I believe that there is some advantage in beginning in the (relatively) concrete context of modules and, consequently, in the first part of the book, modules are simply sets with structure and most of the action takes place in the category of modules. Many results are presented, or at least surveyed, in that first part, so a reader may refer to these without having to absorb the possibly unfamiliar functorial point of view.

Nevertheless, it was convincingly demonstrated by Herzog in the early 1990s that the most efficient and natural way to prove Ziegler’s results, and many subsequent ones, is to move to the appropriate functor category. Indeed it was already appreciated that work, particularly of Gruson and Jensen, ran, in places, parallel to pre-Ziegler results in the model theory of modules and some of the translation between the two languages (model-theoretic and functorial) was already known.

Furthermore, many applications have been to the representation theory of finite-dimensional algebras, where functorial methods have become quite pervasive. So, at the beginning of Part II, we move to the functor category. In fact, the ground will have been prepared already, in the sense that I call on results from Part II in more than a few proofs in Part I. The main reason for this anticipation, and consequent complication in the structure of logical dependencies in the book, is that I wish to present the functorial proofs of many of the basic results. The original model-theoretic and/or algebraic (“non-functorial”) proofs are available elsewhere, whereas the functorial proofs are scattered in the literature and, in many cases, have not appeared.

In the second part of the book modules become certain types of functors on module categories: in the third part they become functors on functor categories. This third part deals with results and questions clustering around relationships between definable categories. Model theory reappears more explicitly in this part because it is, from one point of view, all about interpretability of one category in another.

One could say that Part I is set mostly in the category  $\text{Mod-}R$ , that Part II is set in the functor category  $(R\text{-mod}, \mathbf{Ab})$  and that Part III is set in the category  $((R\text{-mod}, \mathbf{Ab})^{\text{fp}}, \mathbf{Ab})$ . The category,  $\text{Ex}((R\text{-mod}, \mathbf{Ab})^{\text{fp}}, \mathbf{Ab})$ , of exact functors on the category  $(R\text{-mod}, \mathbf{Ab})^{\text{fp}}$  appears and reappears in many forms throughout the book.

A second spectrum also appears. The Zariski spectrum is well known in the context of commutative rings: it is the space of prime ideals endowed with the Zariski topology. It is also possible to define this space, *à la* Gabriel [202], in terms of the category of modules, namely as the set of isomorphism types of indecomposable injectives endowed with a topology which can be defined in terms of morphisms from finitely presented modules. That definition makes sense in the category of modules over any ring, indeed in any locally finitely presented abelian category. Applied to the already-mentioned functor category  $(R\text{-mod}, \mathbf{Ab})$ , one obtains what I call the Gabriel–Zariski spectrum. It turns out that this space can also be obtained from the Ziegler spectrum, as the “dual” topology which has, for a basis of open sets, the complements of compact open sets of the Ziegler topology.

Much less has been done with this than with the Ziegler topology and I give it a corresponding amount of space. I do, however, suspect that there is much to discover about it and to do with it. The Gabriel–Zariski spectrum has a much more geometric character than the Ziegler spectrum, in particular it carries a sheaf of rings which generalises the classical structure sheaf: it is yet another “non-commutative geometry”.

Chapters 1–5 and 10–12, minus a few sections, form the core exposition. The results in the first group of chapters are set in the category of modules and lead the reader through pp conditions and purity to the definition and properties of the Ziegler spectrum. The methods used in the proofs change gradually; from elementary linear algebra to making use of functor categories. One of my reasons for writing this book, rather than being content with what was already in the literature, was to present the basic theory using these functorial methods since they have lead to proofs which are often much shorter and more natural than the original ones. The second group of chapters introduces those methods, so the reader of Chapters 1–5 must increasingly become the reader of Chapters 10–12.

Beyond this core, further general topics are presented in Chapters 6 and 14 (rings and sheaves of definable scalars, the Gabriel–Zariski topology) and in

Chapters 7 and 13 (dimensions). Chapter 9, on ideals in  $\text{mod-}R$ , leads naturally to the view of Part II.

Already that gives us a book of some 500 pages, yet there is much more which should be said, not least applications in specific contexts. At least some of that is said in the remaining pages, though often rather briefly. Chapter 8 contains examples and descriptions of Ziegler spectra over various types of ring. Some of the most fruitful development has been in the representation theory of artin algebras (Chapter 15). The theory applies in categories much more general than categories of modules and Chapters 16 and 17 present examples. In these chapters the emphasis is more on setting out the basic ideas and reporting on what has been done, so rather few proofs are given and the reader is referred to the original sources for the full story.

Though the book begins with systems of linear equations, by the time we arrive at Part III we are entering very abstract territory, an additive universe which parallels that of topos theory. Chapter 18 introduces this, though not at great length since this is work in progress and likely not to be in optimal form.

Ziegler's paper was on the model theory of modules and, amongst all this algebraic development, we should not forget the open questions and developments in that subject, so Appendix D is a, very brief, update on the model theory of modules per se.

Beyond this, there is background on model theory in Appendices A and B, as well as general background (Appendix E) and a model theory/functor category theory "dictionary" (Appendix C).

**Relationship with the earlier book and other work** As to the relationship of this book with my earlier one *Model Theory and Modules* [495], this is, in a sense, a sequel but the emphases of the two books are very different. The earlier book covered model-theoretic aspects of modules and related algebraic topics, and it was written from a primarily model-theoretic standpoint. In this book the viewpoint is algebraic and category-theoretic though it is informed by ideas from model theory. No doubt the model theory proved to be an obstacle for some readers of [495] and perhaps the functor-category theory will play a similar role in this book. But I hope that by introducing the functorial ideas gradually through the first part of this book I will have made the path somewhat easier. Readers who have some familiarity with the contents of [495] will find here new results and fresh directions and they will find that the text reflects a great change in viewpoint and expansion of methods that has taken place in the meantime.

The actual overlap between the books is rather smaller than one might expect, given that they are devoted to the same circle of ideas. Part of the reason for this is that many new ideas and results have been produced in the intervening years.

If that were all, then an update would have sufficed. But one of the reasons for my writing this book is to reflect the fundamental shift in viewpoint, the adoption of a functor-category approach, that has taken place in the area. Although, in this context, the languages of model theory and functor-category theory are, in essence and also in many details, equivalent (indeed, I provide, at Appendix C, a dictionary between them) there have proved to be many conceptual advantages in adopting the latter language. Readers familiar with the arguments of [495] or Ziegler's paper, will see here how complex and sometimes apparently ad-hoc arguments become natural and easy in this alternative language. That is not to say that the insights and techniques of model theory have been abandoned. In fact they inform the whole book, although this might not always be apparent. Some model-theoretic ideas have been explicitly retained, for example the notion of pp-type, because we need this concept and because there is no algebraic name for it. On the other hand, there is no need in this book to treat formulas as objects of a formal language, so I refer to them simply as conditions (which is, in any case, how we think of them).

In the more model-theoretic approach there was a conscious adaptation of ideas from module theory (at least on my part, see, for example, [495, p. 173], and, I would guess, also by Garavaglia, see especially [209]), using the heuristic that pp conditions are generalised ring elements, that pp-types are generalised ideals (in their role as annihilators) and that various arguments involving positive quantifier-free types in injective modules extend to pp-types in pure-injective modules. Moving to the functor category has the effect of turning this analogy into literal generalisation. See, for example, the two proofs of 5.1.3 bearing in mind the heuristic that a pp-type is a generalised right ideal.

Various papers and books contain significant exposition of some of the material included here. Apart from my earlier book, [495], and some papers, [493], [497], [503], [511], there are Rothmaler's, [620], [622], [623], the survey articles [61], [484], [568], [514], [728], a large part of the book, [323], of Jensen and Lenzing, the monograph, [358], of Krause and sections in the books of Facchini, [183], and Puninski, [564]. There is also the more recent [516] which deals with the model theory of definable additive categories.

In this book I have tried to include at least mention of all recent significant developments but, in contrast to the writing of [495], I have tried, with a degree of success, to restrain my tendency to aim to be encyclopaedic. There are some topics which I just mention here because, although I would have liked to have said more, I do not have the expertise to say anything more useful than what can easily be found already in the literature. These include: the work of Guil Asensio and Herzog developing a theory of purity in Flat- $R$  (see the end of Section 4.6); recent and continuing developments around cotilting modules and cotorsion theories (see



---

sample content of Purity, Spectra and Localisation (Encyclopedia of Mathematics and its Applications, Volume 21)

- [read Daybreak \(Titan Trilogy, Book 3\) for free](#)
- [Dead Man's Hand: An Anthology of the Weird West online](#)
- [click Understanding Manga and Anime](#)
- [download online The Ethics of Suicide: Historical Sources pdf, azw \(kindle\), epub, doc, mobi](#)
- [download online Why We Build: Power and Desire in Architecture book](#)
- [read The Web Designer's Idea Book, Volume 4: Inspiration from the Best Web Design Trends, Themes and Styles here](#)
  
- <http://schroff.de/books/The-Laws-of-Lifetime-Growth.pdf>
- <http://flog.co.id/library/Dead-Man-s-Hand--An-Anthology-of-the-Weird-West.pdf>
- <http://aseasonedman.com/ebooks/At-Home-in-Mitford--The-Mitford-Years--Book-1-.pdf>
- <http://ramazotti.ru/library/Anthropologie-philosophique.pdf>
- <http://flog.co.id/library/Getting-Inside-Your-Head--What-Cognitive-Science-Can-Tell-Us-about-Popular-Culture.pdf>
- <http://musor.ruspb.info/?library/Meet-Christopher-Columbus.pdf>