SECOND EDITION

PARTIAL DIFFERENTIAL EQUATIONS

AN INTRODUCTION

Walter A. Strauss

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PARTIAL DIFFERENTIAL EQUATIONS

AN INTRODUCTION

WALTER A. STRAUSS
Brown University

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PREFACE

Our understanding of the fundamental processes of the natural world is based to a large extent on partial differential equations. Examples are the vibrations of solids, the flow of fluids, the diffusion of chemicals, the spread of heat, the structure of molecules, the interactions of photons and electrons, and the radiation of electromagnetic waves. Partial differential equations also play a central role in modern mathematics, especially in geometry and analysis. The availability of powerful computers is gradually shifting the emphasis in partial differential equations away from the analytical computation of solutions and toward both their numerical analysis and the qualitative theory.

This book provides an introduction to the basic properties of partial differential equations (PDEs) and to the techniques that have proved useful in analyzing them. My purpose is to provide for the student a broad perspective on the subject, to illustrate the rich variety of phenomena encompassed by it, and to impart a working knowledge of the most important techniques of analysis of the solutions of the equations.

One of the most important techniques is the method of separation of variables. Many textbooks heavily emphasize this technique to the point of excluding other points of view. The problem with that approach is that only certain kinds of partial differential equations can be solved by it, whereas others cannot. In this book it plays a very important but not an overriding role. Other texts, which bring in relatively advanced theoretical ideas, require too much mathematical knowledge for the typical undergraduate student. I have tried to minimize the advanced concepts and the mathematical jargon in this book. However, because partial differential equations is a subject at the forefront of research in modern science, I have not hesitated to mention advanced ideas as further topics for the ambitious student to pursue.

This is an undergraduate textbook. It is designed for juniors and seniors who are science, engineering, or mathematics majors. Graduate students, especially in the sciences, could surely learn from it, but it is in no way conceived of as a graduate text.

The main prerequisite is a solid knowledge of calculus, especially multivariate. The other prerequisites are small amounts of ordinary differential equations and of linear algebra, each much less than a semester's worth. However, since the subject of partial differential equations is by its very nature not an easy one, I have recommended to my own students that they should already have taken full courses in these two subjects.

The presentation is based on the following principles. Motivate with physics but then do mathematics. Focus on the three classical equations: All the important ideas can be understood in terms of them. Do one spatial dimension before going on to two and three dimensions with their more complicated geometries. Do problems without boundaries before bringing in boundary conditions. (By the end of Chapter 2, the student will already have an intuitive and analytical understanding of simple wave and diffusion phenomena.) Do not hesitate to present some facts without proofs, but provide the most critical proofs. Provide introductions to a variety of important advanced topics.

There is plenty of material in this book for a year-long course. A quarter course, or a fairly relaxed semester course, would cover the starred sections of Chapters 1 to 6. A more ambitious semester course could supplement the basic starred sections in various ways. The unstarred sections in Chapters 1 to 6 could be covered as desired. A computational emphasis following the starred sections would be provided by the numerical analysis of Chapter 8. To resume separation of variables after Chapter 6, one would take up Chapter 10. For physics majors one could do some combination of Chapters 9, 12, 13, and 14. A traditional course on boundary value problems would cover Chapters 1, 4, 5, 6, and 10.

Each chapter is divided into sections, denoted A.B. An equation numbered (A.B.C) refers to equation (C) in section A.B. A reference to equation (C) refers to the equation in the same section. A similar system is used for numbering theorems and exercises. The references are indicated by brackets, like [AS].

The help of my colleagues is gratefully acknowledged. I especially thank Yue Liu and Brian Loe for their extensive help with the exercises, as well as Costas Dafermos, Bob Glassey, Jerry Goldstein, Manos Grillakis, Yan Guo, Chris Jones, Keith Lewis, Gustavo Perla Menzala, and Bob Seeley for their suggestions and corrections.

Walter A. Strauss

PREFACE TO SECOND EDITION

In the years since the first edition came out, partial differential equations has become yet more prominent, both as a model for scientific theories and within mathematics itself. In this second edition I have added 30 new exercises. Furthermore, this edition is accompanied by a solutions manual that has answers to about half of the exercises worked out in detail. I have added a new section on water waves as well as new material and explanatory comments in many places. Corrections have been made wherever necessary.

I would like to take this opportunity to thank all the people who have pointed out errors in the first edition or made useful suggestions, including Andrew Bernoff, Rustum Choksi, Adrian Constantin, Leonid Dickey, Julio Dix, Craig Evans, A. M. Fink, Robert Glassey, Jerome Goldstein, Leon Greenberg, Chris Hunter, Eva Kallin, Jim Kelliher, Jeng-Eng Lin, Howard Liu, Jeff Nunemacher, Vassilis Papanicolaou, Mary Pugh, Stan Richardson, Stuart Rogers, Paul Sacks, Naoki Saito, Stephen Simons, Catherine Sulem, David Wagner, David Weinberg, and Nick Zakrasek. My warmest thanks go to Julie and Steve Levandosky who, besides being my co-authors on the solutions manual, provided many suggestions and much insight regarding the text itself.

CONTENTS

(The starred sections form the basic part of the book.)

1

WHERE PDEs COME FROM

After thinking about the meaning of a partial differential equation, we will flex our mathematical muscles by solving a few of them. Then we will see how naturally they arise in the physical sciences. The physics will motivate the formulation of boundary conditions and initial conditions.

1.1 WHAT IS A PARTIAL DIFFERENTIAL EQUATION?

The key defining property of a partial differential equation (PDE) is that there is more than one independent variable $x, y, \dots$. There is a dependent variable that is an unknown function of these variables $u(x, y, \dots)$. We will often denote its derivatives by subscripts; thus $\partial u/\partial x = u_x$, and so on. A PDE is an identity that relates the independent variables, the dependent variable u, and the partial derivatives of u. It can be written as

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = F(x, y, u, u_x, u_y) = 0. \tag{1}$$

This is the most general PDE in two independent variables of *first* order. The *order* of an equation is the highest derivative that appears. The most general *second*-order PDE in two independent variables is

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{2}$$

A *solution* of a PDE is a function $u(x, y, \dots)$ that satisfies the equation identically, at least in some region of the $x, y, \dots$ variables.

When solving an ordinary differential equation (ODE), one sometimes reverses the roles of the independent and the dependent variables—for instance, for the separable ODE $\dfrac{du}{dx} = u^3$. For PDEs, the distinction between the independent variables and the dependent variable (the unknown) is always maintained.

Some examples of PDEs (all of which occur in physical theory) are:

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - iu_{xx} = 0$ $(i = \sqrt{-1})$ (quantum mechanics)

Each of these has two independent variables, written either as x and y or as x and t. Examples 1 to 3 have order one; 4, 5, and 8 have order two; 6 has order three; and 7 has order four. Examples 3, 5, and 6 are distinguished from the others in that they are not "linear." We shall now explain this concept.

Linearity means the following. Write the equation in the form $\mathscr{L}u = 0$, where $\mathscr{L}$ is an *operator*. That is, if v is any function, $\mathscr{L}v$ is a new function. For instance, $\mathscr{L} = \partial/\partial x$ is the operator that takes v into its partial derivative v_x. In Example 2, the operator $\mathscr{L}$ is $\mathscr{L} = \partial/\partial x + y\partial/\partial y$. ($\mathscr{L}u = u_x + yu_y$.) The definition we want for linearity is

$$\mathscr{L}(u + v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u \tag{3}$$

for any functions u, v and any constant c. Whenever (3) holds (for all choices of u, v, and c), $\mathscr{L}$ is called *linear operator*. The equation

$$\mathscr{L}u = 0 \tag{4}$$

is called *linear* if $\mathscr{L}$ is a linear operator. Equation (4) is called a *homogeneous linear equation*. The equation

$$\mathscr{L}u = g, \tag{5}$$

where $g \neq 0$ is a given function of the independent variables, is called an *inhomogeneous linear equation*. For instance, the equation

$$(\cos xy^2)u_x - y^2u_y = \tan(x^2 + y^2) \tag{6}$$

is an inhomogeneous linear equation.

As you can easily verify, five of the eight equations above are linear as well as homogeneous. Example 5, on the other hand, is not linear because although $(u + v)_{xx} = u_{xx} + v_{xx}$ and $(u + v)_{tt} = u_{tt} + v_{tt}$ satisfy property (3), the cubic term does not:

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 \neq u^3 + v^3.$$

The advantage of linearity for the equation $\mathscr{L}u = 0$ is that if u and v are both solutions, so is $(u + v)$. If $u_1, \ldots, u_n$ are all solutions, so is any linear combination

$$c_1u_1(x) + \cdots + c_nu_n(x) = \sum_{j=1}^{n} c_ju_j(x) \qquad (c_j = \text{ constants}).$$

(This is sometimes called the superposition principle.) Another consequence of linearity is that if you add a homogeneous solution [a solution of (4)] to an inhomogeneous solution [a solution of (5)], you get an inhomogeneous solution. (Why?) The mathematical structure that deals with linear combinations and linear operators is the vector space. Exercises 5–10 are review problems on vector spaces.

We'll study, almost exclusively, linear systems with constant coefficients. Recall that for ODEs you get linear combinations. The coefficients are the arbitrary constants. *For an ODE of order m, you get m arbitrary constants.*

Let's look at some PDEs.

Example 1.

Find all $u(x, y)$ satisfying the equation $u_{xx} = 0$. Well, we can integrate once to get $u_x =$ constant. But that's not really right since there's another variable y. What we really get is $u_x(x, y) = f(y)$, where $f(y)$ is arbitrary. Do it again to get $u(x, y) = f(y)x + g(y)$. This is the solution formula. Note that *there are two arbitrary* functions *in the solution*. We see this as well in the next two examples. □

Example 2.

Solve the PDE $u_{xx} + u = 0$. Again, it's really an ODE with an extra variable y. We know how to solve the ODE, so the solution is

$$u = f(y)\cos x + g(y)\sin x,$$

where again $f(y)$ and $g(y)$ are two arbitrary functions of y. You can easily check this formula by differentiating twice to verify that $u_{xx} = -u$. □

Example 3.

Solve the PDE $u_{xy} = 0$. This isn't too hard either. First let's integrate in x, regarding y as fixed. So we get

$$u_y(x, y) = f(y).$$

Next let's integrate in y regarding x as fixed. We get the solution

$$u(x, y) = F(y) + G(x),$$

where $F' = f$. □

Moral *A PDE has arbitrary functions in its solution.* In these examples the arbitrary functions are functions of one variable that combine to produce a function $u(x, y)$ of two variables which is only partly arbitrary.

A function of two variables contains *immensely* more information than a function of only one variable. Geometrically, it is obvious that a surface $\{u = f(x, y)\}$, the graph of a function of two variables, is a much more complicated object than a curve $\{u = f(x)\}$, the graph of a function of one variable.

To illustrate this, we can ask how a computer would record a function $u = f(x)$. Suppose that we choose 100 points to describe it using equally spaced values of x: $x_1, x_2, x_3, \ldots, x_{100}$. We could write them down in a column, and next to each x_j we could write the corresponding value $u_j = f(x_j)$. Now how about a function $u = f(x, y)$? Suppose that we choose 100 equally spaced values of x and also of y: $x_1, x_2, x_3, \ldots, x_{100}$ and $y_1, y_2, y_3, \ldots, y_{100}$. Each *pair* x_i, y_j provides a value $u_{ij} = f(x_i, y_j)$, so there will be $100^2 = 10{,}000$ lines of the form

$$x_i \qquad y_j \qquad u_{ij}$$

required to describe the function! (If we had a prearranged system, we would need to record only the values u_{ij}.) A function of three variables described discretely by 100 values in each variable would require a million numbers!

To understand this book what do you have to know from calculus? Certainly all the basic facts about partial derivatives and multiple integrals. For a brief discussion of such topics, see the Appendix. Here are a few things to keep in mind, some of which may be new to you.

1. Derivatives are *local*. For instance, to calculate the derivative $(\partial u/\partial x)(x_0, t_0)$ at a particular point, you need to know just the values of $u(x, t_0)$ for x *near* x_0, since the derivative is the limit as $x \to x_0$.
2. Mixed derivatives are equal: $u_{xy} = u_{yx}$. (We assume throughout this book, unless stated otherwise, that all derivatives exist and are continuous.)
3. The chain rule is used frequently in PDEs; for instance,
$$\frac{\partial}{\partial x}[f(g(x, t))] = f'(g(x, t)) \cdot \frac{\partial g}{\partial x}(x, t).$$
4. For the integrals of derivatives, the reader should learn or review Green's theorem and the divergence theorem. (See the end of Section A.3 in the Appendix.)
5. Derivatives of integrals like $I(t) = \int_{a(t)}^{b(t)} f(x, t)\, dx$ (see Section A.3).
6. Jacobians (change of variable in a double integral) (see Section A.1).
7. Infinite series of functions and their differentiation (see Section A.2).
8. Directional derivatives (see Section A.1).
9. We'll often reduce PDEs to ODEs, so we must know how to solve simple ODEs. But we won't need to know anything about tricky ODEs.

EXERCISES

1. Verify the linearity and nonlinearity of the eight examples of PDEs given in the text, by checking whether or not equations (3) are valid.
2. Which of the following operators are linear?
   (a) $\mathscr{L}u = u_x + xu_y$
   (b) $\mathscr{L}u = u_x + uu_y$
   (c) $\mathscr{L}u = u_x + u_y^2$
   (d) $\mathscr{L}u = u_x + u_y + 1$
   (e) $\mathscr{L}u = \sqrt{1+x^2}\,(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$
3. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.
   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x(1+u_x^2)^{-1/2} + u_y(1+u_y^2)^{-1/2} = 0$
   (g) $u_x + e^y u_y = 0$
   (h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$
4. Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.
5. Which of the following collections of 3-vectors $[a, b, c]$ are vector spaces? Provide reasons.
   (a) The vectors with $b = 0$.
   (b) The vectors with $b = 1$.
   (c) The vectors with $ab = 0$.
   (d) All the linear combinations of the two vectors [1, 1, 0] and [2, 0, 1].
   (e) All the vectors such that $c - a = 2b$.
6. Are the three vectors [1, 2, 3], [−2, 0, 1], and [1, 10, 17] linearly dependent or independent? Do they span all vectors or not?
7. Are the functions $1 + x$, $1 - x$, and $1 + x + x^2$ linearly dependent or independent? Why?
8. Find a vector that, together with the vectors [1, 1, 1] and [1, 2, 1], forms a basis of $\mathbb{R}^3$.
9. Show that the functions $(c_1 + c_2 \sin^2 x + c_3 \cos^2 x)$ form a vector space. Find a basis of it. What is its dimension?
10. Show that the solutions of the differential equation $u''' - 3u'' + 4u = 0$ form a vector space. Find a basis of it.
11. Verify that $u(x, y) = f(x)g(y)$ is a solution of the PDE $uu_{xy} = u_x u_y$ for all pairs of (differentiable) functions f and g of one variable.

12. Verify by direct substitution that

$$u_n(x, y) = \sin nx \sinh ny$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

1.2 FIRST-ORDER LINEAR EQUATIONS

We begin our study of PDEs by solving some simple ones. The solution is quite geometric in spirit.

The simplest possible PDE is $\partial u/\partial x = 0$ [where $u = u(x, y)$]. Its general solution is $u = f(y)$, where f is any function of *one* variable. For instance, $u = y^2 - y$ and $u = e^y \cos y$ are two solutions. Because the solutions don't depend on x, they are constant on the lines $y =$ constant in the xy plane.

THE CONSTANT COEFFICIENT EQUATION

Let us solve

$$\boxed{au_x + bu_y = 0,} \tag{1}$$

where a and b are constants not both zero.

Geometric Method The quantity $au_x + bu_y$ is the directional derivative of u in the direction of the vector $\mathbf{V} = (a, b) = a\mathbf{i} + b\mathbf{j}$. It must always be zero. This means that $u(x, y)$ must be constant in the direction of $\mathbf{V}$. The vector $(b, -a)$ is orthogonal to $\mathbf{V}$. The lines parallel to $\mathbf{V}$ (see Figure 1) have the equations $bx - ay =$ constant. (They are called the *characteristic lines*.) The solution is constant on each such line. Therefore, $u(x, y)$ depends on $bx - ay$ only. Thus the solution is

$$\boxed{u(x, y) = f(bx - ay),} \tag{2}$$

where f is any function of one variable. Let's explain this conclusion more explicitly. On the line $bx - ay = c$, the solution u has a constant value. Call

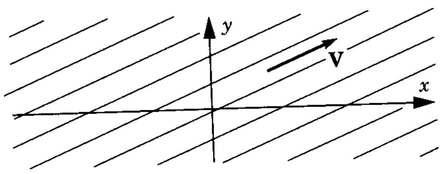

Figure 1

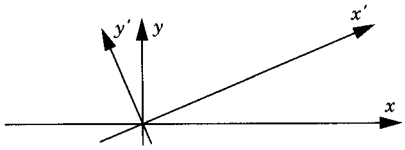

Figure 2

this value $f(c)$. Then $u(x, y) = f(c) = f(bx - ay)$. Since c is arbitrary, we have formula (2) for all values of x and y. In xyu space the solution defines a surface that is made up of parallel horizontal straight lines like a sheet of corrugated iron.

Coordinate Method Change variables (or "make a change of coordinates"; Figure 2) to

$$x' = ax + by \qquad y' = bx - ay. \tag{3}$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. So, since $a^2 + b^2 \neq 0$, the equation takes the form $u_{x'} = 0$ in the new (primed) variables. Thus the solution is $u = f(y') = f(bx - ay)$, with f an arbitrary function of *one* variable. This is exactly the same answer as before!

Example 1.

Solve the PDE $4u_x - 3u_y = 0$, together with the auxiliary condition that $u(0, y) = y^3$. By (2) we have $u(x, y) = f(-3x - 4y)$. This is the general solution of the PDE. Setting $x = 0$ yields the equation $y^3 = f(-4y)$. Letting $w = -4y$ yields $f(w) = -w^3/64$. Therefore, $u(x, y) = (3x + 4y)^3/64$.

Solutions can usually be checked much easier than they can be derived. We check this solution by simple differentiation: $u_x = 9(3x + 4y)^2/64$ and $u_y = 12(3x + 4y)^2/64$ so that $4u_x - 3u_y = 0$. Furthermore, $u(0, y) = (3 \cdot 0 + 4y)^3/64 = y^3$. □

THE VARIABLE COEFFICIENT EQUATION

The equation

$$\boxed{u_x + yu_y = 0} \tag{4}$$

is linear and homogeneous but has a variable coefficient (y). We shall illustrate for equation (4) how to use the geometric method somewhat like Example 1.

The PDE (4) itself asserts that *the directional derivative in the direction of the vector* $(1, y)$ *is zero*. The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y (see Figure 3). Their equations are

$$\frac{dy}{dx} = \frac{y}{1} \tag{5}$$

This ODE has the solutions

$$y = Ce^x. \tag{6}$$

These curves are called the *characteristic curves* of the PDE (4). As C is changed, the curves fill out the xy plane perfectly without intersecting. On each of the curves $u(x, y)$ is a constant because

$$\frac{d}{dx}u(x, Ce^x) = \frac{\partial u}{\partial x} + Ce^x\frac{\partial u}{\partial y} = u_x + yu_y = 0.$$

Thus $u(x, Ce^x) = u(0, Ce^0) = u(0, C)$ is independent of x. Putting $y = Ce^x$ and $C = e^{-x}y$, we have

$$u(x, y) = u(0,\ e^{-x}y).$$

It follows that

$$\boxed{u(x, y) = f(e^{-x}y)} \tag{7}$$

is the *general solution* of this PDE, where again f is an arbitrary function of only a single variable. This is easily checked by differentiation using the chain rule (see Exercise 4). Geometrically, the "picture" of the solution $u(x, y)$ is that it is *constant on each characteristic curve* in Figure 3.

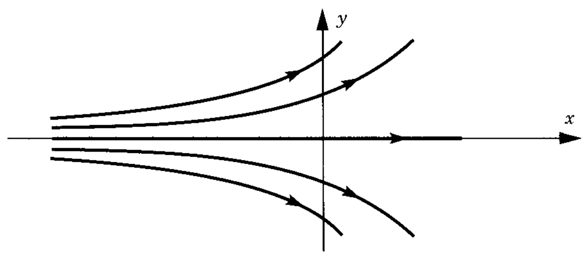

Figure 3

Example 2.

Find the solution of (4) that satisfies the auxiliary condition $u(0, y) = y^3$. Indeed, putting $x = 0$ in (7), we get $y^3 = f(e^{-0}y)$, so that $f(y) = y^3$. Therefore, $u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3$. □

Example 3.

Solve the PDE

$$\boxed{u_x + 2xy^2u_y = 0.} \tag{8}$$

The characteristic curves satisfy the ODE $dy/dx = 2xy^2/1 = 2xy^2$. To solve the ODE, we separate variables: $dy/y^2 = 2x\,dx$; hence $-1/y = x^2 - C$, so that

$$y = (C - x^2)^{-1}. \tag{9}$$

These curves are the characteristics. Again, $u(x, y)$ is a constant on each such curve. (Check it by writing it out.) So $u(x, y) = f(C)$, where f is an arbitrary function. Therefore, the general solution of (8) is obtained by solving (9) for C. That is,

$$\boxed{u(x, y) = f\left(x^2 + \frac{1}{y}\right).} \tag{10}$$

Again this is easily checked by differentiation, using the chain rule: $u_x = 2x \cdot f'(x^2 + 1/y)$ and $u_y = -(1/y^2) \cdot f'(x^2 + 1/y)$, whence $u_x + 2xy^2u_y = 0$. □

In summary, the geometric method works nicely for any PDE of the form $a(x, y)u_x + b(x, y)u_y = 0$. It reduces the solution of the PDE to the solution of the ODE $dy/dx = b(x, y)/a(x, y)$. If the ODE can be solved, so can the PDE. Every solution of the PDE is constant on the solution curves of the ODE.

Moral Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called *initial* or *boundary* conditions. We shall encounter these conditions throughout the book.

EXERCISES

1. Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.
2. Solve the equation $3u_y + u_{xy} = 0$. (*Hint*: Let $v = u_y$.)

3. Solve the equation $(1+x^2)u_x + u_y = 0$. Sketch some of the characteristic curves.
4. Check that (7) indeed solves (4).
5. Solve the equation $xu_x + yu_y = 0$.
6. Solve the equation $\sqrt{1-x^2}u_x + u_y = 0$ with the condition $u(0, y) = y$.
7. (a) Solve the equation $yu_x + xu_y = 0$ with $u(0, y) = e^{-y^2}$.
   (b) In which region of the xy plane is the solution uniquely determined?
8. Solve $au_x + bu_y + cu = 0$.
9. Solve the equation $u_x + u_y = 1$.
10. Solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.
11. Solve $au_x + bu_y = f(x, y)$, where $f(x, y)$ is a given function. If $a \neq 0$, write the solution in the form

$$u(x, y) = (a^2 + b^2)^{-1/2} \int_L f\, ds + g(bx - ay),$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x, y), and the integral is a line integral. (*Hint:* Use the coordinate method.)
12. Show that the new coordinate axes defined by (3) are orthogonal.
13. Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

1.3 FLOWS, VIBRATIONS, AND DIFFUSIONS

The subject of PDEs was practically a branch of physics until the twentieth century. In this section we present a series of examples of PDEs as they occur in physics. They provide the basic motivation for all the PDE problems we study in the rest of the book. We shall see that most often in physical problems the independent variables are those of space x, y, z, and time t.

Example 1. Simple Transport

Consider a fluid, water, say, flowing at a constant rate c along a horizontal pipe of fixed cross section in the positive x direction. A substance, say a pollutant, is suspended in the water. Let $u(x, t)$ be its concentration in grams/centimeter at time t. Then

$$\boxed{u_t + cu_x = 0.} \qquad (1)$$

(That is, the rate of change u_t of concentration is proportional to the gradient u_x. Diffusion is assumed to be negligible.) Solving this equation as in Section 1.2, we find that the concentration is a function of $(x - ct)$

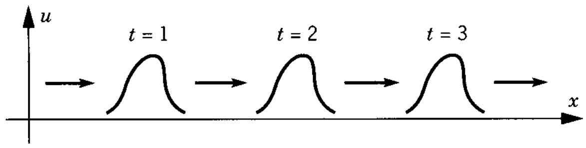

Figure 1

only. This means that the substance is transported to the right at a fixed speed c. Each individual particle moves to the right at speed c; that is, in the xt plane, it moves precisely along a characteristic line (see Figure 1). □

Derivation of Equation (1). The amount of pollutant in the interval $[0, b]$ at the time t is $M = \int_0^b u(x, t)\, dx$, in grams, say. At the later time $t + h$, the same molecules of pollutant have moved to the right by $c \cdot h$ centimeters. Hence

$$M = \int_0^b u(x, t) dx = \int_{ch}^{b+ch} u(x, t + h)\, dx.$$

Differentiating with respect to b, we get

$$u(b, t) = u(b + ch, t + h).$$

Differentiating with respect to h and putting $h = 0$, we get

$$0 = cu_x(b, t) + u_t(b, t),$$

which is equation (1). □

Example 2. Vibrating String

Consider a flexible, elastic homogenous string or thread of length l, which undergoes relatively small transverse vibrations. For instance, it could be a guitar string or a plucked violin string. At a given instant t, the string might look as shown in Figure 2. Assume that it remains in a plane. Let $u(x, t)$ be its displacement from equilibrium position at time t and position x. Because the string is perfectly flexible, the tension (force) is directed tangentially along the string (Figure 3). Let $T(x, t)$ be the magnitude of this tension vector. Let ρ be the density (mass per unit length) of the string. It is a constant because the string is homogeneous. We shall write down Newton's law for the part of the string between any two points at $x = x_0$ and $x = x_1$. The slope of the string at x_1 is

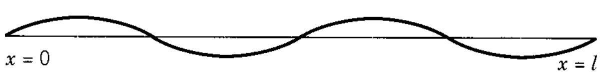

Figure 2

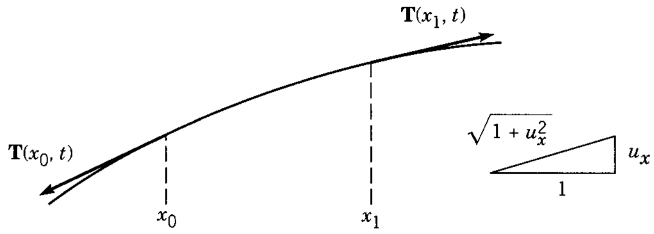

Figure 3

$u_x(x_1, t)$. Newton's law $\mathbf{F} = m\mathbf{a}$ in its longitudinal (x) and transverse (u) components is

$$\left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = 0 \qquad \text{(longitudinal)}$$

$$\left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} \rho u_{tt}\, dx \quad \text{(transverse)}$$

The right sides are the components of the mass times the acceleration integrated over the piece of string. Since we have assumed that the motion is purely transverse, there is no longitudinal motion.

Now we also assume that the motion is small—more specifically, that $|u_x|$ is quite small. Then $\sqrt{1+u_x^2}$ may be replaced by 1. This is justified by the Taylor expansion, actually the binomial expansion,

$$\sqrt{1+u_x^2} = 1 + \tfrac{1}{2}u_x^2 + \cdots$$

where the dots represent higher powers of u_x. If u_x is small, it makes sense to drop the even smaller quantity u_x^2 and its higher powers. With the square roots replaced by 1, the first equation then says that T is constant along the string. Let us assume that T is independent of t as well as x. The second equation, differentiated, says that

$$(Tu_x)_x = \rho u_{tt}.$$

That is,

$$\boxed{u_{tt} = c^2 u_{xx} \quad \text{where } c = \sqrt{\frac{T}{\rho}}.} \tag{2}$$

This is the *wave equation*. At this point it is not clear why c is defined in this manner, but shortly we'll see that c is the *wave speed*. □