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William A. Adkins
Mark G. Davidson

Ordinary Differential Equations

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Preface

This text is intended for the introductory three- or four-hour one-semester sophomore level differential equations course traditionally taken by students majoring in science or engineering. The prerequisite is the standard course in elementary calculus.

Engineering students frequently take a course on and use the Laplace transform as an essential tool in their studies. In most differential equations texts, the Laplace transform is presented, usually toward the end of the text, as an alternative method for the solution of constant coefficient linear differential equations, with particular emphasis on discontinuous or impulsive forcing functions. Because of its placement at the end of the course, this important concept is not as fully assimilated as one might hope for continued applications in the engineering curriculum. *Thus, a goal of the present text is to present the Laplace transform early in the text, and use it as a tool for motivating and developing much of the remaining differential equation concepts for which it is particularly well suited*.

There are several rewards for investing in an early development of the Laplace transform. The standard solution methods for constant coefficient linear differential equations are immediate and simplified. We are able to provide a proof of the existence and uniqueness theorems which are not usually given in introductory texts. The solution method for constant coefficient linear systems is streamlined, and we avoid having to introduce the notion of a defective or nondefective matrix or develop generalized eigenvectors. Even the Cayley–Hamilton theorem, used in Sect. 9.6, is a simple consequence of the Laplace transform. In short, the Laplace transform is an effective tool with surprisingly diverse applications.

Mathematicians are well aware of the importance of transform methods to simplify mathematical problems. For example, the Fourier transform is extremely important and has extensive use in more advanced mathematics courses. The wavelet transform has received much attention from both engineers and mathematicians recently. It has been applied to problems in signal analysis, storage and transmission of data, and data compression. We believe that students should be introduced to transform methods early on in their studies and to that end, the Laplace transform is particularly well suited for a sophomore level course in differential equations. It has been our experience that by introducing the Laplace transform near the beginning of the text, students become proficient in its use and comfortable with this important concept, while at the same time learning the standard topics in differential equations.

Chapter 1 is a conventional introductory chapter that includes solution techniques for the most commonly used first order differential equations, namely, separable and linear equations, and some substitutions that reduce other equations to one of these. There are also the Picard approximation algorithm and a description, without proof, of an existence and uniqueness theorem for first order equations.

Chapter 2 starts immediately with the introduction of the Laplace transform as an integral operator that turns a differential equation in t into an algebraic equation in another variable s. A few basic calculations then allow one to start solving some differential equations of order greater than one. The rest of this chapter develops the necessary theory to be able to efficiently use the Laplace transform. Some proofs, such as the injectivity of the Laplace transform, are delegated to an appendix. Sections 2.6 and 2.7 introduce the basic function spaces that are used to describe the solution spaces of constant coefficient linear homogeneous differential equations.

With the Laplace transform in hand, Chap. 3 efficiently develops the basic theory for constant coefficient linear differential equations of order 2. For example, the homogeneous equation $q(\boldsymbol{D})y = 0$ has the solution space $\mathcal{E}_q$ that has already been described in Sect. 2.6. The Laplace transform immediately gives a very easy procedure for finding the test function when teaching the method of undetermined coefficients. Thus, it is unnecessary to develop a rule-based procedure or the annihilator method that is common in many texts.

Chapter 4 extends the basic theory developed in Chap. 3 to higher order equations. All of the basic concepts and procedures naturally extend. If desired, one can simultaneously introduce the higher order equations as Chap. 3 is developed or very briefly mention the differences following Chap. 3.

Chapter 5 introduces some of the theory for second order linear differential equations that are not constant coefficient. Reduction of order and variation of parameters are topics that are included here, while Sect. 5.4 uses the Laplace transform to transform certain second order nonconstant coefficient linear differential equations into first order linear differential equations that can then be solved by the techniques described in Chap. 1.

We have broken up the main theory of the Laplace transform into two parts for simplicity. Thus, the material in Chap. 2 only uses continuous input functions, while in Chap. 6 we return to develop the theory of the Laplace transform for discontinuous functions, most notably, the step functions and functions with jump discontinuities that can be expressed in terms of step functions in a natural way. The Dirac delta function and differential equations that use the delta function are also developed here. The Laplace transform works very well as a tool for solving such differential equations. Sections 6.6–6.8 are a rather extensive treatment of periodic functions, their Laplace transform theory, and constant coefficient linear differential equations with periodic input function. These sections make for a good supplemental project for a motivated student.

Chapter 7 is an introduction to power series methods for linear differential equations. As a nice application of the Frobenius method, explicit Laplace inversion formulas involving rational functions with denominators that are powers of an irreducible quadratic are derived.

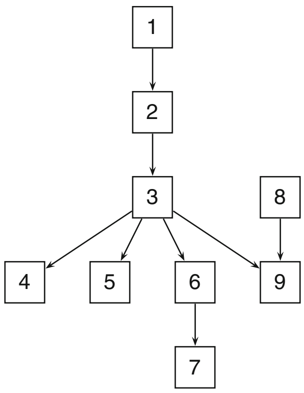
Chapter 8 is primarily included for completeness. It is a standard introduction to some matrix algebra that is needed for systems of linear differential equations. For those who have already had exposure to this basic algebra, it can be safely skipped or given as supplemental reading.

Chapter 9 is concerned with solving systems of linear differential equations. By the use of the Laplace transform, it is possible to give an explicit formula for the matrix exponential $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$ that does not involve the use of eigenvectors or generalized eigenvectors. Moreover, we are then able to develop an efficient method for computing e^{At} known as Fulmer's method. Another thing which is somewhat unique is that we use the matrix exponential in order to solve a constant coefficient system $y' = Ay + f(t)$, $y(t_0) = y_0$ by means of an integrating factor. An immediate consequence of this is the existence and uniqueness theorem for higher order constant coefficient linear differential equations, a fact that is not commonly proved in texts at this level.

The text has numerous exercises, with answers to most odd-numbered exercises in the appendix. Additionally, a student solutions manual is available with solutions to most odd-numbered problems, and an instructors solution manual includes solutions to most exercises.

Chapter Dependence

The following diagram illustrates interdependence among the chapters.



Suggested Syllabi

The following table suggests two possible syllabi for one semester courses.

3-Hour Course	*4-Hour Course*	*Further Reading*
Sections 1.1–1.6	Sections 1.1–1.7	
Sections 2.1–2.8	Sections 2.1–2.8	
Sections 3.1–3.6	Sections 3.1–3.7	
Sections 4.1–4.3	Sections 4.1–4.4	Section 4.5
Sections 5.1–5.3, 5.6	Sections 5.1–5.6	
Sections 6.1–6.5	Sections 6.1–6.5	Sections 6.6–6.8
	Sections 7.1–7.3	Section 7.4
Sections 9.1–9.5	Sections 9.1–9.5, 9.7	Section 9.6
		Sections A.1, A.5

Chapter 8 is on matrix operations. It is not included in the syllabi given above since some of this material is sometimes covered by courses that precede differential equations. Instructors should decide what material needs to be covered for their students. The sections in the Further Reading column are written at a more advanced level. They may be used to challenge exceptional students.

We routinely provide a basic table of Laplace transforms, such as Tables 2.6 and 2.7, for use by students during exams.

Acknowledgments

We would like to express our gratitude to the many people who have helped to bring this text to its finish. We thank Frank Neubrander who suggested making the Laplace transform have a more central role in the development of the subject. We thank the many instructors who used preliminary versions of the text and gave valuable suggestions for its improvement. They include Yuri Antipov, Scott Baldridge, Blaise Bourdin, Guoli Ding, Charles Egedy, Hui Kuo, Robert Lipton, Michael Malisoff, Phuc Nguyen, Richard Oberlin, Gestur Olafsson, Boris Rubin, Li-Yeng Sung, Michael Tom, Terrie White, and Shijun Zheng. We thank Thomas Davidson for proofreading many of the solutions. Finally, we thank the many many students who patiently used versions of the text during its development.

Baton Rouge, Louisiana

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Chapter 1
First Order Differential Equations

1.1 An Introduction to Differential Equations

Many problems of science and engineering require the description of some measurable quantity (position, temperature, population, concentration, electric current, etc.) as a function of time. Frequently, the scientific laws governing such quantities are best expressed as equations that involve the rate at which that quantity changes over time. Such laws give rise to differential equations. Consider the following three examples:

Example 1 (Newton's Law of Heating and Cooling). Suppose we are interested in the temperature of an object (e.g., a cup of hot coffee) that sits in an environment (e.g., a room) or space (called, ambient space) that is maintained at a constant temperature T_{a}. ***Newton's law of heating and cooling*** states that the ***rate*** at which the temperature $T(t)$of the object changes is ***proportional*** to the ***temperature difference between*** the object and ambient space. Since rate of change of $T(t)$ is expressed mathematically as the derivative, $T'(t)$,[1] Newton's law of heating and cooling is formulated as the mathematical expression

$$T'(t) = r(T(t) - T_{\mathrm{a}}),$$

where r is the constant of proportionality. Notice that this is an equation that relates the first derivative $T'(t)$ and the function $T(t)$ itself. It is an example of a differential equation. We will study this example in detail in Sect. 1.3.

Example 2 (Radioactive decay). Radioactive results from the instability of the nucleus of certain atoms from which various particles are emitted. The atoms then

[1] In this text, we will generally use the prime notation, that is, y', y'', y''' (and $y^{(n)}$ for derivatives of order greater than 3) to denote derivatives, but the Leibniz notation $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$, etc. will also be used when convenient.

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decay into other isotopes or even other atoms. *The **law of radioactive decay** states that the **rate** at which the radioactive atoms disintegrate is **proportional** to the total **number of radioactive atoms** present*. If $N(t)$ represents the number of radioactive atoms at time t, then the rate of change of $N(t)$ is expressed as the derivative $N'(t)$. Thus, the law of radioactive decay is expressed as the equation

$$N'(t) = -\lambda N(t).$$

As in the previous example, this is an equation that relates the first derivative $N'(t)$ and the function $N(t)$ itself, and hence is a differential equation. We will consider it further in Sect. 1.3.

As a third example, consider the following:

Example 3 (Newton's Laws of Motion). Suppose $s(t)$ is a position function of some body with mass m as measured from some fixed origin. We assume that as time passes, forces are applied to the body so that it moves along some line. Its velocity is given by the first derivative, $s'(t)$, and its acceleration is given by the second derivative, $s''(t)$. ***Newton's second law of motion** states that the **net force** acting on the body is the **product of its mass and acceleration***. Thus,

$$ms''(t) = F_{\text{net}}(t).$$

Now in many circumstances, the net force acting on the body depends on time, the object's position, and its velocity. Thus, $F_{\text{net}}(t) = F(t, s(t), s'(t))$, and this leads to the equation

$$ms''(t) = F(t, s(t), s'(t)).$$

A precise formula for F depends on the circumstances of the given problem. For example, the motion of a body in a spring-body-dashpot system is given by $ms''(t) + \mu s'(t) + ks(t) = f(t)$, where μ and k are constants related to the spring and dashpot and $f(t)$ is some applied external (possibly) time-dependent force. We will study this example in Sect. 3.6. For now though, we just note that this equation relates the second derivative to the function, its derivative, and time. It too is an example of a differential equation.

Each of these examples illustrates two important points:

- Scientific laws regarding physical quantities are frequently expressed and best understood in terms of how that quantity changes.
- The mathematical model that expresses those changes gives rise to equations that involve derivatives of the quantity, that is, differential equations.

We now give a more formal definition of the types of equations we will be studying. An ***ordinary differential equation*** is an equation relating an unknown function $y(t)$, some of the derivatives of $y(t)$, and the variable t, which in many applied problems will represent time. The domain of the unknown function is some interval of the real line, which we will frequently denote by the symbol I.[2] The ***order*** of a differential equation is the order of the highest derivative that appears in the differential equation. Thus, the order of the differential equations given in the above examples is summarized in the following table:

Differential equation	Order
$T'(t) = r(T(t) - T_{\mathrm{a}})$	1
$N'(t) = -\lambda N(t)$	1
$ms''(t) = F(t, s(t), s'(t))$	2

Note that $y(t)$ is our generic name for an unknown function, but in concrete cases, the unknown function may have a different name, such as $T(t)$, $N(t)$, or $s(t)$ in the examples above. The ***standard form*** for an ordinary differential equation is obtained by solving for the highest order derivative as a function of the unknown function $y = y(t)$, its lower order derivatives, and the independent variable t. Thus, a first order ordinary differential equation is expressed in standard form as

$$y'(t) = F(t,\, y(t)), \tag{1}$$

a second order ordinary differential equation in standard form is written

$$y''(t) = F(t,\, y(t),\, y'(t)), \tag{2}$$

and an nth order differential equation is expressed in standard form as

$$y^{(n)}(t) = F(t,\, y(t),\, \ldots,\, y^{(n-1)}(t)). \tag{3}$$

The standard form is simply a convenient way to be able to talk about various hypotheses to put on an equation to insure a particular conclusion, such as *existence and uniqueness of solutions* (discussed in Sect. 1.7) and to classify various types of equations (as we do in this chapter, for example) so that you will know which algorithm to apply to arrive at a solution. In the examples given above, the equations

$$\begin{aligned} T'(t) &= r(T(t) - T_{\mathrm{a}}), \\ N'(t) &= -\lambda N(t) \end{aligned}$$

are in standard form while the equation in Example 3 is not. However, simply dividing by m gives

$$s''(t) = \frac{1}{m} F(t,\, s(t),\, s'(t)),$$

a second order differential equation in standard form.

[2] Recall that the standard notations from calculus used to describe an interval I are $(a,\, b)$, $[a,\, b)$, $(a,\, b]$, and $[a,\, b]$ where $a < b$ are real numbers. There are also the infinite length intervals $(-\infty,\, a)$ and $(a,\, \infty)$ where a is a real number or $\pm\infty$.

In differential equations involving the unknown function $y(t)$, the variable t is frequently referred to as the ***independent variable***, while y is referred to as the ***dependent variable***, indicating that y has a functional dependence on t. In writing ordinary differential equations, it is conventional to suppress the implicit functional evaluations $y(t)$, $y'(t)$, etc. and write y, y', etc. Thus the differential equations in our examples above would be written

$$T' = r(T - T_{\mathrm{a}}),$$
$$N' = -\lambda N,$$
$$\text{and} \quad s'' = \frac{1}{m} F(t, s, s'),$$

where the dependent variables are respectively, T, N, and s.

Sometimes we must deal with functions $u = u(t_1, t_2, \ldots, t_n)$ of two or more variables. In this case, a ***partial differential equation*** is an equation relating u, some of the partial derivatives of u with respect to the variables $t_1, \ldots, t_n$, and possibly the variables themselves. While there may be a time or two where we need to consider a partial differential equation, the focus of this text is on the study of ordinary differential equations. Thus, when we use the term differential equation without a qualifying adjective, you should assume that we mean *ordinary* differential equation.

Example 4. Consider the following differential equations. Determine their order, whether ordinary or partial, and the standard form where appropriate:

1. $y' = 2y$
2. $y' - y = t$
3. $y'' + \sin y = 0$
4. $y^{(4)} - y'' = y$
5. $ay'' + by' + cy = A\cos\omega t \quad (a \neq 0)$
6. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

► **Solution.** Equations (1)–(5) are ordinary differential equations while (6) is a partial differential equation. Equations (1) and (2) are first order, (3) and (5) are second order, and (4) is fourth order. Equation (1) is in standard form. The standard forms for (2)–(5) are as follows:

2. $y' = y + t$
3. $y'' = -\sin y$
4. $y^{(4)} = y'' + y$
5. $y'' = -\dfrac{b}{a}y' - \dfrac{c}{a}y + \dfrac{A}{a}\cos\omega t$ ◄

Solutions

In contrast to algebraic equations, where the given and unknown objects are numbers, differential equations belong to the much wider class of ***functional***

equations in which the given and unknown objects are functions (scalar functions or vector functions) defined on some interval. A ***solution of an ordinary differential equation*** is a function $y(t)$ defined on some specific interval $I \subseteq \mathbb{R}$ such that substituting $y(t)$ for y and substituting $y'(t)$ for y', $y''(t)$ for y'', etc. in the equation gives a ***functional identity***. That is, an identity which is satisfied for *all* $t \in I$. For example, if a first order differential equation is given in standard form as $y' = F(t, y)$, then a function $y(t)$ defined on an interval I is a solution if

$$y'(t) = F(t, y(t)) \quad \text{for all } t \in I.$$

More generally, $y(t)$, defined on an interval I, is a solution of an nth order differential equation expressed in standard form by $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$ provided

$$y^{(n)}(t) = F(t, y(t), \dots, y^{(n-1)}(t)) \quad \text{for all } t \in I.$$

It should be noted that it is not necessary to express the given differential equation in standard form in order to check that a function is a solution. Simply substitute $y(t)$ and the derivatives of $y(t)$ into the differential equation as it is given. The ***general solution*** of a differential equation is the set of all solutions. As the following examples will show, writing down the general solution to a differential equation can range from easy to difficult.

Example 5. Consider the differential equation

$$y' = y - t. \tag{4}$$

Determine which of the following functions defined on the interval $(-\infty, \infty)$ are solutions:

1. $y_1(t) = t + 1$
2. $y_2(t) = e^t$
3. $y_3(t) = t + 1 - 7e^t$
4. $y_4(t) = t + 1 + ce^t$ where c is an arbitrary scalar.

► **Solution.** In each case, we calculate the derivative and substitute the results in (4). The following table summarizes the needed calculations:

Function	$y'(t)$	$y(t) - t$
$y_1(t) = t + 1$	$y_1'(t) = 1$	$y_1(t) - t = t + 1 - t = 1$
$y_2(t) = e^t$	$y_2'(t) = e^t$	$y_2(t) - t = e^t - t$
$y_3(t) = t + 1 - 7e^t$	$y_3'(t) = 1 - 7e^t$	$y_3(t) - t = t + 1 - 7e^t - t = 1 - 7e^t$
$y_4(t) = t + 1 + ce^t$	$y_4'(t) = 1 + ce^t$	$y_4(t) - t = t + 1 + ce^t - t = 1 + ce^t$

For $y_i(t)$ to be a solution of (4), the second and third entries in the row for $y_i(t)$ must be the same. Thus, $y_1(t)$, $y_3(t)$, and $y_4(t)$ are solutions while $y_2(t)$ is not a

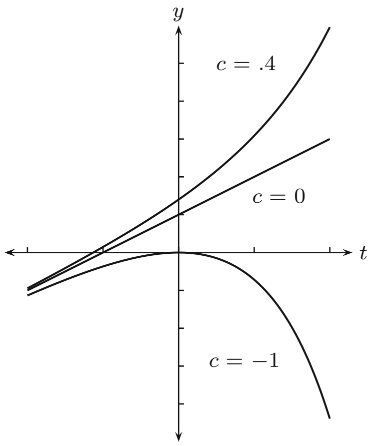

Fig. 1.1 The solutions $y_g(t) = t + 1 + ce^t$ of $y' = y - t$ for various c

solution. Notice that $y_1(t) = y_4(t)$ when $c = 0$ and $y_3(t) = y_4(t)$ when $c = -7$. Thus, $y_4(t)$ actually already contains $y_1(t)$ and $y_3(t)$ by appropriate choices of the constant $c \in \mathbb{R}$, the real numbers. ◀

The differential equation given by (4) is an example of a first order *linear* differential equation. The theory of such equations will be discussed in Sect. 1.4, where we will show that *all* solutions to (4) are included in the function

$$y_4(t) = t + 1 + ce^t, \qquad t \in (-\infty, \infty)$$

of the above example by appropriate choice of the constant c. We call this the general solution of (4) and denote it by $y_g(t)$. Figure 1.1 is the graph of $y_g(t)$ for various choices of the constant c.

Observe that the general solution is parameterized by the constant c, so that there is a solution for each value of c and hence there are infinitely many solutions of (4). This is characteristic of many differential equations. Moreover, the domain is the same for each of the solutions, namely, the entire real line. With the following example, there is a completely different behavior with regard to the domain of the solutions. Specifically, the domain of each solution varies with the parameter c and is not the same interval for all solutions.

Example 6. Consider the differential equation

$$y' = -2t(1 + y)^2. \tag{5}$$

Show that the following functions are solutions:

1. $y_1(t) = -1$
2. $y_2(t) = -1 + (t^2 - c)^{-1}$, for any constant c

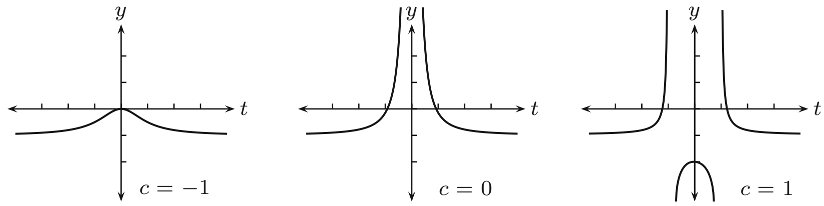

Fig. 1.2 The solutions $y_2(t) = -1 + (t^2 - c)^{-1}$ of $y' = -2t(1 + y)^2$ for various c

► **Solution.** Let $y_1(t) = -1$. Then $y_1'(t) = 0$ and $-2t(1 + y_1(t))^2 = -2t(0) = 0$, which is valid for all $t \in (-\infty, \infty)$. Hence, $y_1(t) = -1$ is a solution.

Now let $y_2(t) = -1 + (t^2 - c)^{-1}$. Straightforward calculations give

$$y_2'(t) = -2t(t^2 - c)^{-2}, \text{ and}$$
$$-2t(1 + y_2(t))^2 = -2t(1 + (-1 + (t^2 - c)^{-1}))^2 = -2t(t^2 - c)^{-2}.$$

Thus, $y_2'(t) = -2t(1 + y_2(t))^2$ so that $y_2(t)$ is a solution for any choice of the constant c. ◄

Equation (5) is an example of a *separable* differential equation. The theory of separable equations will be discussed in Sect. 1.3. It turns out that there are no solutions to (5) other than $y_1(t)$ and $y_2(t)$, so that these two sets of functions constitute the general solution $y_g(t)$. Notice that the intervals on which $y_2(t)$ is defined depend on the constant c. For example, if $c < 0$, then $y_2(t) = -1 + (t^2 - c)^{-1}$ is defined for all $t \in (-\infty, \infty)$. If $c = 0$, then $y_2(t) = -1 + t^{-2}$ is defined on two intervals: $t \in (-\infty, 0)$ or $t \in (0, \infty)$. Finally, if $c > 0$, then $y_2(t)$ is defined on three intervals: $(-\infty, -\sqrt{c})$, $(-\sqrt{c}, \sqrt{c})$, or $(\sqrt{c}, \infty)$. Figure 1.2 gives the graph of $y_2(t)$ for various choices of the constant c.

Note that the interval on which the solution $y(t)$ is defined is not at all apparent from looking at the differential equation (5).

Example 7. Consider the differential equation

$$y'' + 16y = 0. \tag{6}$$

Show that the following functions are solutions on the entire real line:

1. $y_1(t) = \cos 4t$
2. $y_2(t) = \sin 4t$
3. $y_3(t) = c_1 \cos 4t + c_2 \sin 4t$, where c_1 and c_2 are constants.

Show that the following functions are not solutions:

4. $y_4(t) = e^{4t}$
5. $y_5(t) = \sin t$.

► **Solution.** In standard form, (6) can be written as $y'' = -16y$, so for $y(t)$ to be a solution of this equation means that $y''(t) = -16y(t)$ for all real numbers t. The following calculations then verify the claims for the functions $y_i(t)$, $(1 \le i \le 5)$:

1. $y_1''(t) = \dfrac{d^2}{dt^2}(\cos 4t) = \dfrac{d}{dt}(-4\sin 4t) = -16\cos 4t = -16y_1(t)$
2. $y_2''(t) = \dfrac{d^2}{dt^2}(\sin 4t) = \dfrac{d}{dt}(4\cos 4t) = -16\sin 4t = -16y_2(t)$
3. $y_3''(t) = \dfrac{d^2}{dt^2}(c_1\cos 4t + c_2\sin 4t) = \dfrac{d}{dt}(-4c_1\sin 4t + 4c_2\cos 4t)$
$$= -16c_1\cos 4t - 16c_2\sin 4t = -16y_3(t)$$
4. $y_4''(t) = \dfrac{d^2}{dt^2}(e^{4t}) = \dfrac{d}{dt}(4e^{4t}) = 16e^{4t} \ne -16y_4(t)$
5. $y_5''(t) = \dfrac{d^2}{dt^2}(\sin t) = \dfrac{d}{dt}(\cos t) = -\sin t \ne -16y_5(t)$ ◄

It is true, but not obvious, that letting c_1 and c_2 vary over all real numbers in $y_3(t) = c_1\cos 4t + c_2\sin 4t$ produces all solutions to $y'' + 16y = 0$, so that $y_3(t)$ is the general solution of (6) . This differential equation is an example of a second order *constant coefficient linear* differential equation. These equations will be studied in Chap. 3.

The Arbitrary Constants

In Examples 5 and 6, we saw that the solution set of the given first order equation was parameterized by an arbitrary constant c (although (5) also had an extra solution $y_1(t) = -1$), and in Example 7, the solution set of the second order equation was parameterized by two constants c_1 and c_2. To understand why these results are not surprising, consider what is arguably the simplest of all first order differential equations:

$$y' = f(t),$$

where $f(t)$ is some continuous function on some interval I. Integration of both sides produces a solution

$$y(t) = \int f(t)\,dt + c, \tag{7}$$

where c is a constant of integration and $\int f(t)\,dt$ is any fixed antiderivative of $f(t)$. The fundamental theorem of calculus implies that all antiderivatives are of this form so (7) is the general solution of $y' = f(t)$. Generally speaking, solving any first order differential equation will implicitly involve integration. A similar calculation