MODULAR MATHEMATICS

Ordinary Differential Equations

W Cox

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W Cox

Computer Science and Applied Mathematics, Aston University

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Dedication

In memory of my mother, Gwendoline

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Elsevier Ltd.
Linacre House, Jordan Hill, Oxford OX2 8DP
200 Wheeler Road, Burlington, MA 01803
Transferred to digital printing 2004

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library
ISBN 0 340 63203 8
Typeset in 10/12 Times by
Paston Press Ltd, Loddon, Norfolk

Series Preface

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This series is designed particularly, but not exclusively, for students reading degree programm based on semester-long modules. Each text will cover the essential core of an area of mathematics ar lay the foundation for further study in that area. Some texts may include more material than can b comfortably covered in a single module, the intention there being that the topics to be studied can b selected to meet the needs of the student. Historical contexts, real life situations, and linkages wi other areas of mathematics and more advanced topics are included. Traditional worked examples ar exercises are augmented by more open-ended exercises and tutorial problems suitable for group wo or self-study. Where appropriate, the use of computer packages is encouraged. The first level tex assume only the A-level core curriculum.

Preface

This book describes the theory, solution and application of ordinary differential equations. Dependin on the background of the students it is intended to provide ample material for a twenty-four lectu modular course, although the method of presentation is designed to be equally conducive independent learning. The content may be harmlessly reduced by omitting Chapter 9, which aims set Fourier series (which students may meet elsewhere at the same level) in the context of differenti equations, and/or Chapter 10.

The style of presentation is novel, and is intended to engage the reader at crucial points. Th problems within the text are exercises designed to lead the reader into the subsequent text - it prepar them for what is to come. Some of these are quite simple, while others form substantial pieces of wo which the impatient reader may avoid by continuing with the text. I refer to this approach as ‘ask an answer’ (a question leads you into the answer, encouraging you to discover the ideas for yourself you wish) as opposed to ‘tell and test’ (we tell you in the text, then test what you have learnt in th exercises).

Most sections have reinforcement exercises, and all chapters but the last have end-of-chapt exercises. These include essay type questions, which exercise your appreciation and understanding the material. These may also provide small group discussion topics. Also you are encouraged to wri your own ‘verbal’ summary of each chapter, as concisely as possible, and compare your result wi the Appendix. Writing such summaries is surprisingly instructive, and provides you with an overvie of the topic of the chapter, and the interrelations between its components. While answers to mo exercises are included, you are encouraged to verify your answers for yourself where possible. Th develops confidence and provides plenty of exercise in manipulative skills.

Some sections, and exercises, are ‘projects’. These give you the opportunity to develop material f yourself, by reference to other books such as those in the bibliography, or by your own work. Suc material is optional and provides opportunities for the enthusiastic reader to delve deeper.

Lastly, I have tried to make the material enjoyable. That doesn’t mean it will always be easy!

Bill Cox, *September 1995*

Acknowledgements

My warm thanks to Barry Martin and Chris Collinson for careful reading of the manuscript and mar helpful comments. Also many thanks to Val Tyas for her forbearance in word-processing the man iterations (this is **definitely** the last one), and Graham Smith for the diagrams. Any remaining erro are entirely mine, of course.

Bill Cox, *Computer Science and Applied Mathematics, Aston University*

1

Introduction and a Look Ahead

1.1 Getting an overview

Coming to a new mathematics topic such as differential equations is like entering a new land covere with forests of details, conceptual peaks, rivers and highways of understanding, exquisite hidd valleys of interest, and broad featureless plains of monotony. There are various ways you can explo a new land: a guided tour with an expert; a 'follow the colour coded signposts' forest walk; a leisure cruise along the main highways, taking in the sights; live there for a while, wandering through t highways and byways, climbing the peaks by different routes. But however you do it, before you del into the guide book, you will greatly benefit from a preliminary study of the atlas, to get yo bearings. The problem with this geographical analogy is that whilst we are already familiar with t geographical and topographical terminology needed to describe the features of a new land, this rarely the case in a new mathematics topic – it is doubly difficult to distinguish the wood from t trees if you don't know what either looks like! However, with some imagination we can lay out t broad features of a new topic using already familiar ideas – and that's what we will do in this chapte relying only on very elementary knowledge of calculus and differential equations.

The study of differential equations can be split into a number of broad areas

- Notation and terminology
  - What is the order of a differential equation?
  - What is a 'solution' to a differential equation?
- Analytical aspects
  - Existence theorems – under what conditions does a solution exist?
  - Uniqueness theorems – when is a solution unique?
- Methods of solution
  - Range of techniques and their applicability. Exact and approximate methods.
- Applications
  - Constructing and solving models with differential equations.

Depending on your interests, some of these areas may be more relevant than others to you. F example the student of pure mathematics might reasonably be expected to use and prove an existen theorem for a particular class of equations. An engineer on the other hand might be more interested solution methods (particularly numerical methods).

The remaining sections of this chapter enable you to develop an overview of the subject matter a structure of the rest of the book, whatever your interests. The individual chapters will give the detai and provide opportunity to hone the skills which you need.

1.2 Notation, terminology and analytical aspects

Before we can start to discuss differential equations there is a certain amount of notation a terminology to deal with. The **order** and **degree** of ordinary differential equations are used to classi equations, usually for the purposes of analysing or solving them, There is a major division betwe **linear** and **nonlinear** equations. With a few simple exceptions, nonlinear equations are impossible t solve except by approximate or numerical methods. Analytical questions such as existence an uniqueness are similarly difficult to address for nonlinear equations. However, a great deal may b learnt about nonlinear equations by limiting our ambitions and looking at general **qualitati** behaviour of solutions. Such qualitative methods, which rely heavily on differential geometry, a briefly discussed in Chapter 10. For the most part the rest of this book is concerned with line equations.

There is a large amount of theoretical knowledge relating to linear equations, as well as man different solution methods. Perhaps the central idea is that of the principle of superposition solutions, which enables us to construct further solutions of linear equations from linear combinatio of known solutions. Another powerful general principle is the construction of the general solution an inhomogeneous ordinary differential equation from the general solution of the associat homogeneous equation and a particular solution of the inhomogeneous equation ('complementa function' + 'particular integral'). This is a general principle applicable to all inhomogeneous line equations, not just-differential equations, and is a fundamental result in the theory of line transformations. These, and other, general principles are important both at the theoretical and th practical level. Thus, the split into 'complementary function' and 'particular integral' often reduc the solution of an inhomogeneous equation to little more than guesswork.

By 'analytical aspects' we mean such questions as existence and uniqueness theorems – the 'pu mathematical analysis', as opposed to the specific methods or techniques of solution. Existence an uniqueness theorems abound for linear equations. Such theorems tell us the conditions under which particular type of differential equation actually has a solution, and to what extent it is unique. The analytical aspects of ordinary differential equations are sometimes regarded as not 'practical', or rather peripheral 'abstract' mathematics. Nothing could be further from the truth. An existenc theorem can at least tell you whether it is worth your while looking for a solution to a give differential equation. A uniqueness theorem can tell you under what conditions a solution, once foun is the **only** solution. This means that you can guess a solution (and you may be surprised how much logical 'deductive' mathematics is in fact educated guesswork), safe in the knowledge that if you ca verify that it is a solution then it must be **the** solution.

1.3 Direct methods of solution

Large oaks from little acorns grow.

PROBLEM 1

Solve this acorn

$$\mathbf{y}' = \lambda \mathbf{y} \quad \mathbf{y} > 0 \qquad \textbf{(1.1)}$$

If the solution is not already second nature to you, then you probably used separation of variables

$$\int \frac{dy}{y} = \lambda \int dx$$

So

$$\ln y = \lambda x + A \quad A \text{ arbitary}$$

isan **implicit** form of the general solution, which may be written **explicitly** as

$$y = Ce^{\lambda x} \quad C = e^{A}$$

Ofcourse, not all first-order equations can be solved in this way. For example the general first-order linear equation

$$y' = P(x)y + Q(x)$$

cannotbe solved immediately by separation of variables – but it can be integrated directly using an integrating factor which is found by separation of variables. The general message here is that new equations can often be solved by rearranging them in order to take advantage of ones you have already solved.

Back to the acorn. The linear second-order equation

$$y'' - 3y' + 2y = 0$$

reallyonly differs from (1.1) by a second derivative term. But one derivative of an exponential function looks pretty much like another so maybe a solution like $y = e^{\lambda x}$ will work again.

PROBLEM 2

Try a function of the form $y = e^{\lambda x}$ in this second-order equation and find the values of λ for which this is a solution. Write down the most general form of the solution stating the principle you have used. How do you know that the resulting solution obtained by such guesswork is the general solution – how do we know there are not others lurking about, unknown to us?

Substituting $y = e^{\lambda x}$ in the equation gives

$$(\lambda^2 - 3\lambda + 2)e^{\lambda x} = 0$$

This can only be an identity, true for all values of x, if

$$\lambda^2 - 3\lambda + 2 = 0$$

i.e. λ can be 1 or 2.

So two possible solutions are $y = e^x, e^{2x}$. Since the equation is linear and homogeneous, any linear combination of these solutions is also a solution, allowing us to construct a solution of the form

$$y = Ae^x + Be^{2x}$$

where A and B are arbitrary constants. The occurrence of two arbitrary constants in this solution might suggest to us that it **is** the general solution – but we would need an appropriate existence and uniqueness theorem for second-order linear equations to confirm this for us.

This short plausible treatment (referred to as 'heuristic'), with which you may be familiar, actually conceals a tremendous amount of theoretical assumption and background – the linear superposition, the treatment of e^x, e^{2x} as somehow independent functions, and finally the implicit use of a **uniqueness theorem** to assure us that the final form of the solution, containing two arbitrary constants, is the unique solution. But all that really concerns us here is that the acorn has sprouted shoots already, enabling us to deal with second- and indeed higher-order linear homogeneous equations.

But what about inhomogeneous equations like

$$y' - y = x + 1\,?$$

You might already be able to solve this by an integrating factor, obtaining

$$y = Ce^x - (x + 2)$$

with C an arbitrary constant.

But let's anticipate a general result, which you may have met in the context of second-order equations, and at the same time take advantage of Problem 1. Assume, as we will prove in Section 2.2, that the general solution of such an inhomogeneous equation is the sum of the general solution (**complementary function**) of the homogeneous equation

$$y' - y = 0$$

and a particular solution (**particular integral**) of the full inhomogeneous equation. Problem 1 gives us the complementary function

$$y_c = Ce^x$$

so we only have to find a particular solution. The acorn helps here too, in that it has alerted us to the possibilities of **guessing** a solution (mathematicians make such guesswork respectable by expressions such as ‘by inspection’, ‘choosing an *ansatz*’ etc., but they all amount basically to inspired guesswork). Maybe we can guess a particular integral. Perhaps something like the right-hand side will work?

PROBLEM 3

Try substituting $y_p = Lx + M$ in the equation, where L and M are constants, and choose L, M to yield a solution.

We find

$$L - (Lx + M) \equiv x + 1$$

from which $L = -1$, $M = -2$, so

$$y_p = -(x + 2)$$

isthe desired particular integral. The general solution is thus

$$y = y_c + y_p = C\mathrm{e}^x - (x + 2)$$

asfound by the integrating factor method.

The lesson to learn here is – to find a particular solution to an inhomogeneous equation try a solution of similar form to the right-hand side, although this will not always work. All this, and a lot more to come, we owe to playful extrapolation from the acorn. But also note that we have again used a definite notion of a solution of a differential equation (resulting in the identity from which L and M are determined), and also relied on a uniqueness theorem to reassure us that the solution obtained by guesswork is indeed the solution.

EXERCISES ON 1.3

1. Solve the equation

$$xy' = (3x - 1)y \quad x \neq 0$$

by putting $z = xy$.

2. Solve the equation

$$y'' - 5y' + 6y = 0$$

andcheck your answer by substituting into the equation.

1.4 Transform methods

We can already see the importance of the exponential function in linear differential equations. Yo almost wonder if we could somehow change the function *y(x)*, by some sort of exponenti transformation, to a new function which simplifies the whole business of dealing with differenti equations. One way of transforming functions such as *y(x)* is by an **integral transform** and, given th importance of the exponential function, it is perhaps no surprise that a transform of the form

$$\mathcal{L}[y(x)] = \tilde{y}(s) = \int_0^\infty \mathrm{e}^{-sx} y(x)dx \quad s > 0$$

doesprove useful in differential equations. This is in fact the **Laplace transform**, which we study detail in Chapter 6.

PROBLEM 4

Prove that $\mathcal{L}[y'] = s\tilde{y}(s) - y(0)$.

We have, from the above definition,

$$\begin{aligned}\mathcal{L}[y'] &= \int_0^\infty \mathrm{e}^{-sx} \frac{dy(x)}{dx} dx \\ &= [y(x)\mathrm{e}^{-sx}]_0^\infty + s\int_0^\infty e^{-sx} y(x)dx \\ &= -y(0) + s\tilde{y}(s)\end{aligned}$$

on integrating by parts and simplifying the ‘boundary term’. (We will fill in the details in Chapter 6.)

The point is that this transform converts the derivative to simple multiplication by *s* – and automatically incorporates the initial value *y*(0), i.e. it reduces a problem in calculus to one in algebra. This suggests it may be useful in the solution of initial value problems in which *y* is specified at *x* = 0. Let’s see how it deals with the acorn.

PROBLEM 5

Solve the initial value problem

$$\mathbf{y' = y \quad y(0) = 1}$$

by taking the Laplace transform of the equation.

We have

$$\mathcal{L}[y' - y] = \mathcal{L}[y'] - \mathcal{L}[y] = s\tilde{y}(s) - y(0) - \tilde{y}(s) = \mathcal{L}[0] = 0$$

So

$$(s-1)\tilde{y}(s) = y(0) = 1 \quad \tilde{y}(s) = \frac{1}{s-1} \ .$$

Thus the solution *y(x)* has Laplace transform $\tilde{y}(s) = 1/(s-1)$. As you will see in Chapter 6 this yields

$$y(x) = e^x$$

Inthis case the Laplace transform looks like a sledgehammer to crack an acorn, but in more complicated initial value problems the Laplace transform is a powerful tool and cracks much harder nuts.

EXERCISE ON 1.4

Solve the initial value problem

$$y' = y + 1 \quad y(0) = 1$$

by Laplace transform. *Hint*. Find the Laplace transform of 1. Split $\tilde{y}(s)$ into partial fractions.

1.5 Solution in series

(1.1) has already produced a sizeable tree – but surely we have exhausted the theme – there must b
differential equations not owing something to such a simple equation? What about

$$(1-x^2)y'' - 2xy' + 2y = 0$$

whichis linear, but has non-constant coefficients? It is the **Legendre equation**, which we study Chapter 8. We are not going to solve it here, but we can use the acorn to get an insight into how w can solve it. The key lies in what we understand by the function e^x. If I asked you for a definition of I suspect you would give its series definition

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Thissuggests yet another way to solve the acorn.

PROBLEM 6

Solve (1.1) by assuming a power series solution of the form $y = a_0 + a_1x + a_2x^2 + \ldots$We have

$$y' = a_1 + 2a_2x + 3a_3x^2 + \dot{c}\,\dot{c}\,\dot{c} = y = a_0 + a_1x + a_2x^2 + \ldots$$

Since this must be true for all values of x we find, by equating coefficients of like powers of x, that

$$na_n = a_{n-1} \quad n = 1, 2, 3, \ldots$$

Solvingthis **recurrence relation** gives

$$a_n = \frac{a_0}{n!}$$

soa solution of the differential equation is

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) = a_0e^x$$

wherea_0 is arbitrary. Clearly, this method can be applied to many other types of equation to obtain series solutions, including the Legendre equation given above. It is merely fortuitous that in the case of the acorn we actually know that the series represents the function e^x. We now have another method for solving differential equations – the **series solution** method – to which Chapter 8 is devoted.

EXERCISE ON 1.5

Assume a power series solution for the equation

$$(1-x)y' - y = 0 \quad |x| < 1$$

and show that the coefficients, a_n, satisfy $a_n = a_{n-1}$. Why is the condition $|x| \le 1$ specified?

Deduce the general solution and compare with the result obtained by separation of variables.

1.6 Systems of equations

What about systems of equations? Consider the system

$$y'_1 = 2y_1 - 3y_2 \quad y'_2 = y_1 - 2y_2$$

Writeit in matrix form

$$\mathbf{y}' = A\mathbf{y}$$

where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Looksfamiliar? Would you be surprised to learn that the solution of this **matrix differential equatio** can be written

$$\mathbf{y} = \mathrm{e}^{Ax}\mathbf{C}$$

where**C** is a constant 2 × 1 vector, and e^{Ax} is the matrix exponential function defined by

$$\mathrm{e}^{Ax} = I + Ax + \frac{A^2x^2}{2!} + \frac{A^3x^3}{3!} + \ldots$$

where$I$ is the 2 × 2 unit matrix? We study such systems of equations in Chapter 7.

There is also another way of dealing with this system if you know a little matrix theory. Consid the matrix

$$U = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

PROBLEM 7

Show that the columns of *U* are the eigenvectors of *A*.
Show that

$$U^{-1}AU = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv D$$

Rewrite the matrix differential equation for *y* in the form

$$\frac{\mathrm{d}(U^{-1}\mathbf{y})}{\mathrm{d}x} \equiv \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}x} = U^{-1}AUU^{-1}\mathbf{y} \equiv D\mathbf{Y}$$

to obtain the system

$$Y'_1 = Y_1 \quad Y'_2 = -Y_2$$

i.e. two acorns!

This method is the **diagonalization** method, which essentially diagonalizes the coefficient matrix to decouple the system of differential equations and reduce it to a simpler system. It may be applied to any number of linear equations, and also to inhomogeneous systems.

EXERCISE ON 1.6

Solve the equations obtained for Y_1, Y_2 and hence obtain expressions for the solutions y_1, y_2 for the original system of this section.

1.7 Qualitative methods

Suppose we couldn't solve the acorn – that is, there was no simple function like $y = e^x$ which satisfie it. Then we can still say quite a lot about the solution simply by studying the differential equatic itself.

First, note that multiplying (1.1) through by an arbitrary constant:

$$C\frac{dy}{dx} = \frac{d(Cy)}{dx} = Cy$$

showsthat if y is a solution then so is Cy. Intuitively, we feel that the form of the solution curve y $y(x)$ should be fixed once we specify an initial point through which it passes. Indeed, since repeate differentiation of the equation will furnish us with all required derivatives, once such an initial poi is specified, we can see that we could generate the Taylor series for the solution $y = y(x)$ near to Since, therefore, the solution is (at least locally) determined by the single initial condition, we expe the general solution $y = y(x)$ to contain one arbitrary parameter only. We have seen above that if y is solution, so is Cy with C arbitrary, and so the solution can be written in the form

$$y = Cf(x)$$

where, from what we have said above, $f(x)$ must be a function of x only, containing no further arbitra constants. It also tells us that if y is a solution then so is $-y$. The solution curves are therefo symmetric about the x-axis and we can confine ourselves to $y > 0$.

Next, note that the gradient of any solution curve at the point (x, y) is always equal to y for a values of x. In particular, as y gets smaller, the slope of the curve does too, and as y increases, so do the slope of the curve. This tells us that for $y > 0$ any solution curve is monotonic increasing, and h no turning points. If we were to plot the field of gradients of solution curves from the differenti equation, then it would soon become clear that the solution curves have an exponential form. All th is apparent without actually solving the equation.

EXERCISE ON 1.7

Plot the gradient field for (1.1) at selected points and by connecting these by eye construct a sketch of the solution to the initial value problem.

$$y' = y \quad y(0) = 1$$

up to $x = 2$. Your sketch should, of course, approximate to the e_x curve.

1.8 Numerical methods

Qualitative methods can give useful information about the solutions of differential equations eve when we cannot solve them exactly. More detailed information about solutions may be found b numerical methods of a number of different types. The gradient field qualitative approach of th previous section can be refined into such a method. Essentially, one starts from a fixed point on particular solution curve and uses the gradient, calculated from dy/dx via the differential equation, take us to a neighbouring point near the solution curve. The process is then repeated. Again, (1. provides a simple example of this. Thus, suppose we wish to solve the initial value problem

$$y' = y \quad y(0) = 1 \ .$$

PROBLEM 8

Starting from x = 0, obtain an approximation for y(0.1) by approximating the slope of the tangent to the solution curve at x = 0 by

$$\frac{\mathbf{y}(0.1) - \mathbf{y}(0)}{0.1 - 0}$$

This is approximately equal to dy/dx at x = 0, which is given, from the differential equation, by $y(0) = 1$. So

$$\frac{y(0.1) - y(0)}{0.1} \simeq y(0) = 1$$

$$y(0.1) \simeq y(0) + 0.1y(0) \simeq 1 + 0.1 = 1.1$$

Thus, an approximation to the solution of the initial value problem at x = 0.1 is 1.1. The actual value is $e^{0.1} \cong 1.1052$, so our approximation is not bad.

PROBLEM 9

Now repeat the process starting from x = 0.1, using the approximation for y(0.1) found above, to obtain an approximation for y(0.2)

We have

$$\frac{y(0.2) - y(0.1)}{0.2 - 0.1} \simeq \frac{dy}{dx}(0.1) = y(0.1) \simeq 1.1$$

So

$$y(0.2) \simeq y(0.1) + 0.1y(0.1) \simeq 1.1 + 0.1 \times 1.1 = 1.21$$

Thusan approximation to y(0.2) is 1.21. The correct value is $e^{0.2} \cong 1.2214$. Again, not bad.

PROBLEM 10

Repeat this process to get approximations to the solution for x= 0.3, 0.4, 0.5,