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George Pólya

Robert E. Tarjan

Donald R. Woods

Notes on Introductory Combinatorics

Reprint of the 1983 Edition

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Notes on Introductory Combinatorics

George Pólya
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Donald R. Woods

In the winter of 1978, Professors George Pólya and Robert Tarjan teamed up at Stanford University to teach a course titled "Introduction to Combinatorics". This book consists primarily of the class notes and related material produced by Donald Woods as teaching assistant for the course.

Among the topics covered in the notes are elementary subjects such as combinations and permutations, mathematical tools such as generating functions and Pólya's Theory of Counting, and specific problems such as Ramsey Theory, matchings, and Hamiltonian and Eulerian paths.

PREFACE

In the winter of 1978, Professor George Pólya and I jointly taught Stanford University's introductory combinatorics course. This was a great opportunity for me, as I had known of Professor Pólya since having read his classic book, *How to Solve It*, as a teenager. Working with Pólya, who was over ninety years old at the time, was every bit as rewarding as I had hoped it would be. His creativity, intelligence, warmth and generosity of spirit, and wonderful gift for teaching continue to be an inspiration to me.

Combinatorics is one of the branches of mathematics that play a crucial role in computer science, since digital computers manipulate discrete, finite objects. Combinatorics impinges on computing in two ways. First, the properties of graphs and other combinatorial objects lead directly to algorithms for solving graph-theoretic problems, which have widespread application in non-numerical as well as in numerical computing. Second, combinatorial methods provide many analytical tools that can be used for determining the worst-case and expected performance of computer algorithms. A knowledge of combinatorics will serve the computer scientist well.

Combinatorics can be classified into three types: enumerative, existential, and constructive. Enumerative combinatorics deals with the counting of combinatorial objects. Existential combinatorics studies the existence or nonexistence of combinatorial configurations. Constructive combinatorics deals with methods for actually finding specific configurations (as opposed to merely demonstrating their existence theoretically). The first two-thirds of our course, taught by Professor Pólya, dealt with enumerative combinatorics, including combinations, generating functions, the principle of inclusion and exclusion, Stirling numbers, and Pólya's own theory of counting. The last third of the course, taught by me, covered existential combinatorics, with an emphasis on algorithmic graph theory, and included matching, network flow, Hamiltonian and Eulerian paths, and planar graphs.

Donald Woods, our teaching assistant, was not only invaluable in helping us give the course but also was able to prepare readable and comprehensive course notes, which he has edited to form the present book. Don did a masterful job in making sense out of our ramblings and adding observations and references of his own. Were I to teach the course again these notes would be indispensable. I hope you will enjoy them.

Robert E. Tarjan
Murray Hill, New Jersey
May 3, 1983

Table of Contents

1 Introduction

For the most part the notes that comprise this text differ only slightly from those provided to the students during the course. The notes have been merged into a single paper, a few sections have been made more detailed, and various corrigenda have been incorporated. The midterm and final examinations are included in their proper chronological places within the text (chapters 8 and 15), together with the solutions. The only information omitted from this book is that regarding the mechanics of the course—office hours, grading criteria, etc. Homework assignments are included, as they often led to further discussion in the notes. Lecture dates are included to give a feel for the pace at which material was covered, though it should be noted that much of the material in the notes was not actually presented in the lectures, being instead drawn from supplementary notes provided by the instructors or supplied by the author as the notes were written.

A brief word of explanation regarding the dual instructorship of the course: Professor Pólya taught the first two-thirds of the course, reflected in chapters 2 through 7 of this book. Professor Tarjan taught the remainder of the course, as covered in chapters 9 through 14.

Though there was no formal text for the course, a number of books were made available for reference. These books, along with additional texts used by the author in preparing the notes, are listed in the bibliography. The [bracketed] abbreviations given there will be used when referring to one of Pólya's books; the other texts will be referred to by their authors. Though all of the books contain relevant material, not all are *specifically* referenced in this book. In particular, all mentions of [Harary] refer to *Graph Theory* and not to *A Seminar on Graph Theory*.

The author would like to thank Chris Van Wyk and Jim Boyce for their assistance in preparing the manuscript, Don Knuth and Ron Graham for encouraging the publication of the notes and, of course, Professors Pólya and Tarjan for providing ample source material.

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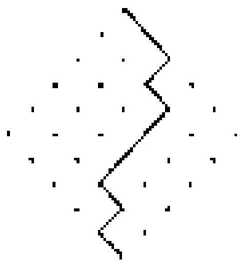
2 Combinations and Permutations

January 5. In his first lecture, Pólya discussed in general terms what combinatorics is about: The study of counting various combinations or configurations. He started with a problem based on the mystical sign known, appropriately, as an "abracadabra".

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```

The question is, how many different ways are there to spell out "abracadabra", always going from one letter to an adjacent letter? Due to the way some letters (especially C and D) are found only in certain rows, it turns out the only ways to spell "abracadabra" start with the topmost 'A' and zig-zag down to the bottommost 'A'. If we think of the letters as points, then any spelling of "abracadabra" specifies a sequence of points forming a crooked line from the top to the bottom. One such line is shown below.



You can also think of this problem in terms of a network of streets in a city where all blocks are the same size. Then the problem becomes one of computing how many ways there are of getting from the northern corner to the southern corner in the minimum number (10) of blocks. (That 10 is the minimum can be seen from the fact that each block, in addition to taking us either east or west, takes us

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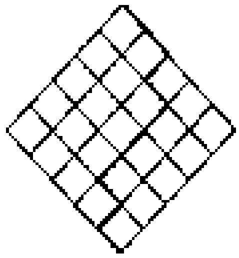
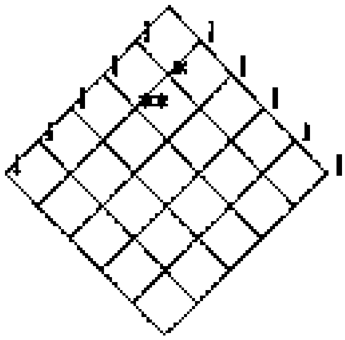
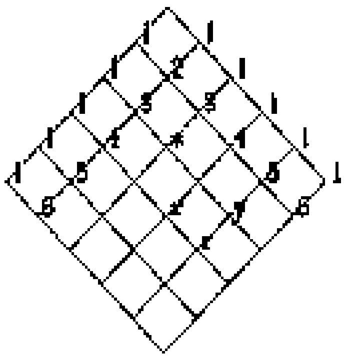
southward one-tenth the t tal southward distance between the two corners.)



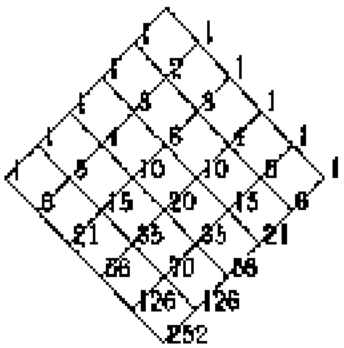
It was decided empirically (*i.e.*, by taking a vote) that there were more than 100 paths, but there was disagreement over whether there were more than 1000, so Pólya proceeded to approach the problem by more formal methods. He began by emphasising an important maxim which you should always consider when working on *any* problem: "*If you cannot solve the proposed problem, solve first a suitable related problem.*" In this instance, the related problem is that of computing how many different paths there are from the northern corner to various other corners, still restricting ourselves to travelling only southeast and southwest. For starters, there is only one path to each of the corners on the northeast edge, namely the path consisting of travelling always southeast and never southwest. Similarly, there is only one path to each of the corners on the northwest edge. We note these values by writing them next to the corners involved.



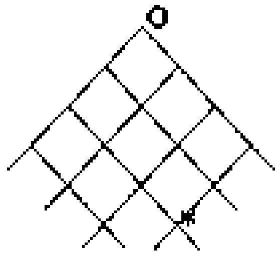
Now what about the corner marked with a *? You could get there by going one block southeast followed by one block southwest, or by going first southwest and then southeast. Similarly, to get to the corner marked **, you could go southeast, then southwest twice, or you could go southwest, then southeast, then southwest, or you could go southwest twice and then southeast. Moving down the diagonal in the manner and, by symmetry, the corresponding diagonal on the eastern side, we can fill in some more values.



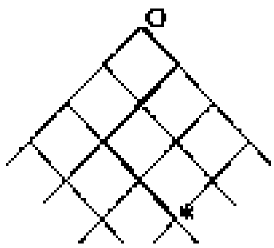
Had we tried to go much further like this, it would probably have gotten rather tiresome, so instead we came up with a general observation regarding an arbitrary corner, such as the one marked z above. If we know that there are x ways to get to the corner just northwest of z, and y ways to get to the corner northeast of z, then there are $x+y$ ways to get to z, since to get there we must first get to either x or y, after which there's only one way to continue on to z. For instance, there are $3+3=6$ paths to the corner marked $*$. This general rule provides us with an easy method to finish computing the number of paths to the southern corner. The first homework assignment was to complete this computation. It comes as no surprise that everyone got it right. For the record, here it is.



The numbers we've been computing are known as binomial coefficients, for reasons we'll get to eventually. The arrangement of the numbers, when cut off by any horizontal line so as to form a triangular pattern, is known as Pascal's triangle. (Pascal referred to it as "the arithmetical triangle".) The numbers are uniquely defined by the boundary condition (the 1's along the edges) together with the recursion formula (each number not on the edge is the sum of the two above it). In addition to this recursion formula, which defines each number in terms of earlier ones, there is another way to look at the situation. Here's a small chunk of the street network we've been working with:



Suppose we want to know the number of different paths (of minimum length) from the origin O to the starred corner. Each such path must consist of 5 blocks, of which exactly 3 go to the right (as seen from above). If we specify which 3 of the 5 blocks will go to the right, we uniquely specify the path. For instance, if we choose the 1st, 4th, and 5th blocks, we get this path:



Conversely, each path from O to * specifies a unique set of 3 blocks that go to the right. So the number of paths is the same as the number of ways of choosing 3 blocks out of the total 5. Euler's notation for this sort of thing is $\binom{5}{3}$ or, in general, $\binom{n}{r}$, denoting the number of ways of choosing a subset of size r from a set of size n.

This is usually read "n-choose-r". (Another name often heard to describe this value, but which recently has fallen out of favor, is that used by Jacob Bernoulli: the combinations of n elements taken r at a time.) Computing this value is the first problem of combinatorics.

Next we come to some basic rules for working with multiple sets. The rules are fairly simple (as basic rules are wont to be), but are nevertheless very important (again as basic rules are wont to be). First off, suppose that out of a set of possibilities, A, it is possible to choose any one of m different elements. From another set, B, it is possible to choose any one of n elements. We wish to select an element from either A or B; we don't care which. *Assuming A and B have no elements in common*, there are $m+n$ possible choices.

Next, suppose the elements of A are $a_1, a_2, \ldots, a_m$, and the elements of B are $b_1, b_2, \ldots, b_n$. We wish to select two elements, one from each set, in a specific order (say, first one from A and then one from B). This operation is known as the Cartesian product of the two sets, due to its relationship with the rectangular (Cartesian) coordinate system. For instance, if A has three elements and B has two, there are six possible pairs: (a_1,b_1), (a_1,b_2), (a_2,b_1), (a_2,b_2), (a_3,b_1), and (a_3,b_2). In general, there are $m \cdot n$ possibilities.

Finally, take a more general case of the Cartesian product. Suppose that, having chosen a_1, we then have a choice among a set of elements $b_{11}, b_{12}, \ldots, b_{1n}$. If we start by choosing a_2, we then have a choice from a *different* set: $b_{21}, b_{22}, \ldots, b_{2n}$, and so on. In general, the possibilities for b differ depending upon our choice for a, *but there are always n of them*. As long as the number of possibilities for b is constant, the total number of pairs (a_i,b_j) is still $m \cdot n$. We'll see an application of this in a moment.

A permutation is an ordering of a set of objects. For instance, given the set of three numbers $\{1,2,3\}$, we could order them in any of 6 different ways: $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, or $\{3,2,1\}$. The number of different permutations of n elements is denoted by P_n. Hence $P_3 = 6$. We also see fairly easily that $P_1 = 1$ and $P_2 = 2$. At this point Pólya stated another important maxim: "*The beginning of most discoveries is to recognise a pattern.*" There is a pattern to the three numbers we've got so far; to make it more apparent, we can rewrite them as follows:

$$P_1 = 1 = 1$$
$$P_2 = 2 = 1 \cdot 2$$
$$P_3 = 6 = 1 \cdot 2 \cdot 3.$$

We conjecture that $P_n = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$. This product is called <u>n factorial</u> and is usually written "$n!$". Now we need to prove our conjecture. Well, *suppose* it's true that $P_n = n!$ Then what would P_{n+1} be? It is the number of ways of ordering $n+1$ objects. The $n+1^{st}$ object could be in any one of $n+1$ positions. Whichever of these positions we choose, the remaining n objects can be ordered in any of P_n ways. Using the generalisation of the Cartesian product rule, we conclude that the total number of ways we can order $n+1$ objects is $(n+1) \cdot P_n$. Therefore, if $P_n = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, then $P_{n+1} = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1) = (n+1)!$. But we know that $P_3 = 3!$, so taking $n=3$ we conclude that $P_4 = 4!$. Knowing this, we can take $n=4$ and conclude that $P_5 = 5!$, and so on. For any finite n, we can prove that $P_n = n!$ by starting at P_3 and chugging away for a while. This method of proof, which Pólya describes as "a diabolic way of proving things", is called <u>mathematical induction</u>. It is extremely useful since it saves you from having to figure out the formula you're proving. If you can make a "lucky guess" as to what the answer is, you may be able to prove it by induction.

January 10. Pólya began the lecture by reviewing the material from the previous lecture. In doing so he brought out some points that hadn't been explicitly stated before. In particular, there's the formal definition of the binomial coefficients:

Boundary condition: $\binom{n}{0} = \binom{n}{n} = 1$

Recursion: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
[n and r integers, $0<r<n+1$]

Similarly, P_n can be defined by boundary conditions and recursion:

Boundary condition: $P_1 = 1! = 1$

Recursion: $P_n = n! = nP_{n-1}$.

If we apply this recursion formula with $n=1$, we find that $P_1 = 1 \cdot P_0$. Hence we define $P_0 = 0! = 1$.

From here, we move on to look at something Pólya called a "variation", a word you may immediately forget. It is defined as follows. Given a set of n objects, we wish to choose r of them <u>in some order</u>. That is, choosing the first object and then the second would be considered different from choosing the second and then the first. How many such variations are there? One approach is to start by choosing some object to be the first one selected. There are n choices. For each choice, there are $n-1$ choices for the second object. Thus, by the product rule, there are $n(n-1)$ choices for the first two objects together. For each such pair, there are $(n-2)$ objects remaining from which to choose the third object. So there are $n(n-1)(n-2)$ choices for the first three objects. Continuing in this manner, we find that there are $n(n-1)(n-2)\ldots(n-r+1)$ variations.

We can often learn something by solving a problem in two different ways, so here's a second approach. We first choose the subset of r objects from among the n. We know there are $\binom{n}{r}$ ways to do this. We then choose the ordering for the r objects. We know how many ways there are to do this, too; it's P_r. So there are $\binom{n}{r} P_r$ variations. But this answer must be the same as the one we got the other way. Therefore $\binom{n}{r} \cdot P_r = n(n-1)(n-2)\ldots(n-r+1)$. So we have learned something new:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot r}.$$

(Note that, in the second form, the sum of 'corresponding' terms in the numerator and denominator is always $n+1$; this can be a useful mnemonic for remembering what the last term in the numerator is.) For example, the number that we computed for the first homework assignment is $\binom{10}{5}$, which by this formula is $(10\cdot9\cdot8\cdot7\cdot6)/(1\cdot2\cdot3\cdot4\cdot5) = (10\cdot9\cdot7\cdot6)/(1\cdot3\cdot5) = (2\cdot9\cdot7\cdot6)/3 = 2\cdot9\cdot7\cdot2 = 252$. It's always a good idea to test out a formula on some special cases where we already know the answer, so let's look at $\binom{n}{n}$ and $\binom{n}{0}$. We have

$$\binom{n}{n} = \frac{n(n-1)(n-2)\ldots1}{1\cdot2\cdot3\cdot\ldots\cdot n},$$

which, since the numerator and denominator have all the same factors, albeit in different orders, indeed equals 1. $\binom{n}{0}$, however, poses a bit of a problem, since the numerator has no factors. By defining the product of zero factors to be equal to 1 (just as $0! = 1$) we find that $\binom{n}{0} = 1$ as expected.

Another way we can get this explicit form for the binomial coefficients is by using mathematical induction. We assume it's true for small n (we can check this by hand) and then show that, if it's true for n, it's true for $n+1$. The first problem on the second homework assignment was to carry out this proof. Here it is: We assume that, for some value of n,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot r}$$

for all values of r. Substituting $r-1$ for r, we find

$$\binom{n}{r-1} = \frac{n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (r-1)}.$$

By the definition of the binomial coefficients, we know that

$$\begin{aligned}
\binom{n+1}{r} &= \binom{n}{r-1} + \binom{n}{r} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (r-1)} + \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot r} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+2) \cdot r}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (r-1) \cdot r} + \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot r} \\
&= \frac{n(n-1)(n-2)\ldots(n-r+2) \cdot (r + n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot (r-1) \cdot r} \\
&= \frac{(n+1)n(n-1)(n-2)\ldots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot r},
\end{aligned}$$

which is the formula we're trying to prove (with $n+1$ substituted for n). Hence, if the formula is true for n, it's true for $n+1$. This, combined with the fact that it's true for $n=1$, means it is true for all finite n. (Actually, there's a minor flaw in this proof. To wit, the recursion formula cannot be used to compute $\binom{n}{n}$ or $\binom{n}{0}$, since it would involve coefficients outside the range $0 \le r \le n$. However, we've already shown separately that these two special cases satisfy the formula, so we're all right.)

A more compact way to write the formula for the binomial coefficient can be derived by multiplying both the numerator and denominator by the factors $(n-r)$, $(n-r-1)$, and so on down to 1.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)(n-r-1)\ldots 2\cdot 1}{(1\cdot 2\cdot 3\cdot\ldots\cdot r)\,(1\cdot 2\cdot\ldots\cdot(n-r-1)\cdot(n-r))}$$
$$= \frac{n!}{r!(n-r)!}$$

Notice that, based on this formula, it is immediately apparent that $\binom{n}{r} = \binom{n}{n-r}$. This was to be expected, since by the method of its construction Pascal's triangle is clearly symmetric.

Next, we consider n houses. They are built identically, because it's easier that way. But then, to make them look different, they are painted different colors: r of them are painted red, s of them yellow, and the remaining t of them green. In how many ways can we assign the colors to the houses? We first choose which houses will be painted red; there are $\binom{n}{r}$ ways to make this choice. Whatever choice we make, there are $n-r$ houses left, of which we choose s to be painted yellow; there are $\binom{n-r}{s}$ ways to do this. At this point we have no choices left to make, since all the rest must be green (that is, $r+s+t=n$). So what do we have? By the product rule, there are $\binom{n}{r}\binom{n-r}{s}$ ways to paint the houses. Using the formula we worked out a moment ago, we find

$$\binom{n}{r}\binom{n-r}{s} = \frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{s!(n-r-s)!}\,.$$

But $n-r-s=t$, and the $(n-r)!$ factors cancel, leaving us with

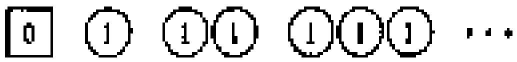
$$\frac{n!}{r!s!t!}\,,$$

which is, fortunately, symmetric with respect to r, s, and t. (The alternative to its being symmetric would be for it to be wrong, since the original problem was symmetric.) This sort of formula is called a <u>multinomial coefficient</u>.

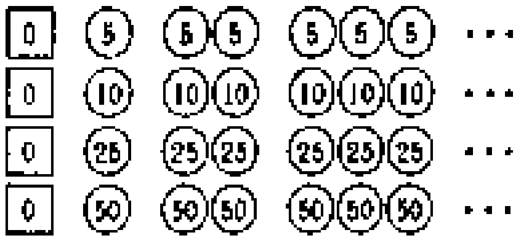
3 Generating Functions

Generating functions are a general mathematical tool developed by de Moivre, Stirling, and Euler in the 18th century, and are used often in combinatorics. As usual, we start by taking a concrete example: In how many ways can you make change for a dollar? We'll assume that we're dealing with only five types of coins—pennies, nickels dimes, quarters, and half dollars.

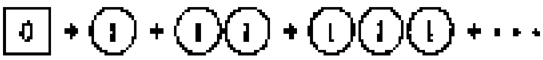
We first consider how many pennies to use. We could use one, or two, or three, etc., and of course we could use none. We can show these choices pictorially:



Similarly, we have an infinite number of choices as to how many nickels we use (although for almost all such choices we'll have more than a dollar already), and how many dimes, and so on:



In giving change for a dollar, or for any other amount, we are effectively choosing exactly one 'heap' from each of the five rows. Within each row, we'll represent the fact that we are choosing a single element by writing the row as a summation:



Next, we represent the combining of the choices from the various rows by writing the product of the rows (the reason for all this will be seen shortly).

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$$
\begin{aligned}
&(\boxed{0} + 1 + 1\,1 + 1\,1\,1 + \cdots)\\
\times\ &(\boxed{0} + 5 + 5\,5 + 5\,5\,5 + \cdots)\\
\times\ &(\boxed{0} + 10 + 10\,10 + 10\,10\,10 + \cdots)\\
\times\ &(\boxed{0} + 25 + 25\,25 + 25\,25\,25 + \cdots)\\
\times\ &(\boxed{0} + 50 + 50\,50 + 50\,50\,50 + \cdots)
\end{aligned}
$$

Now, why did we do this? Well, if we look at the infinite product we've created, we find that each term in the product is the product of five terms, one from each of the sums. Thus, each term of the product corresponds to a different combination of coins, and if we look at *all* the terms of the product, we'll find they include *all* such combinations.

But we don't *want* all combinations; we just want the ones that add up to a dollar. Pólya introduced the symbol x to represent 1¢. So, for example,

$$
\begin{aligned}
1\,1\,1 &= xxx = x^3,\\
5\,5 &= x^5x^5 = x^{10}, \text{ and}\\
\boxed{0} &= x^0 = 1.
\end{aligned}
$$

Our product can now be written more mathematically as follows:

$$
\begin{aligned}
&(1 + x + x^2 + x^3 + \cdots)\\
&\quad\cdot (1 + x^5 + x^{10} + x^{15} + \cdots)\\
&\quad\cdot (1 + x^{10} + x^{20} + x^{30} + \cdots)\\
&\quad\cdot (1 + x^{25} + x^{50} + x^{75} + \cdots)\\
&\quad\cdot (1 + x^{50} + x^{100} + x^{150} + \cdots)
\end{aligned}
$$

For example, one of the terms in the product will be $x^3 \cdot x^5 \cdot x^{20} \cdot 1 \cdot x^{50}$, which corresponds to the combination of coins that consists of 3 pennies, 1 nickel, 2 dimes, no quarters, and 1 half dollar. When the five terms, one from each infinite sum, are multiplied, the exponents add; this is just what we want, because it means the exponent (in our example it's 78) is the total value of the selected coins. So for each combination of coins totalling one dollar, there will be a term in the product with an exponent of 100. If we combine terms that have the same exponent, we get something of the form

$$1 + E_1x + E_2x^2 + \cdots + E_{100}x^{100} + \cdots.$$

January 12. All we need to do is find the coefficient E_{100}. But how do we do that? We could try multiplying out the infinite product, but this would probably take a while. Instead we use what we know about series, and in particular about geometric series.

Consider a typical geometric series: $1 + x + x^2 + x^3 + \cdots$. What does this series sum to? Pólya claimed it was obvious: the sum is S. That doesn't sound like it helps much but, having given it a name, we can manipulate it mathematically. In particular, we can multiply S by $(1-x)$:

$$\begin{aligned} S(1-x) &= 1 + x + x^2 + x^3 + \cdots \\ &\quad\; - x - x^2 - x^3 - x^4 - \cdots \\ &= 1 \end{aligned}$$

So $S = 1/(1-x)$. Similarly, $1 + x^5 + x^{10} + x^{15} + \cdots = 1/(1-x^5)$. Our infinite product thus simplifies to the somewhat more compact form

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})},$$

which we can turn back into a series in powers of x, even though we don't yet know the coefficients numerically, as

$$\sum_{n=0}^{\infty} E_nx^n.$$

Such a summation, in either form, is called the <u>generating function</u> for the sequence $E_0, E_1, E_2, \ldots$.

So far so good, but we don't appear to be any closer to computing E_{100} than we were before. Once again we'll try first solving an easier related problem. In fact, we'll set up a sequence of problems leading to the one we're interested in.

$$\frac{1}{(1-x)} = \sum_{n=0}^{\infty} A_n x^n$$

$$\frac{1}{(1-x)(1-x^5)} = \sum_{n=0}^{\infty} B_n x^n$$

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})} = \sum_{n=0}^{\infty} C_n x^n$$

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})} = \sum_{n=0}^{\infty} D_n x^n$$

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})} = \sum_{n=0}^{\infty} E_n x^n$$

We already know that $A_n = 1$ for all $n \geq 0$. What about B_n? We take the second equation above and multiply both sides by $1-x^5$. The left side becomes $1/(1-x)$, which is series A.

$$\sum_{n=0}^{\infty} A_n x^n = (1-x^5)(\sum_{n=0}^{\infty} B_n x^n)$$

$$= (\sum_{n=0}^{\infty} B_n x^n) - (\sum_{n=0}^{\infty} B_n x^{n+5})$$

What does it mean for these two sums to be equal? Since they must be equal for all values of x, it means that the coefficients of x^n must be equal for all n. On the left side the coefficient is simply A_n. On the right side, the first summation contributes a term of $B_n x^n$, and the second summation contributes $-B_{n-5}x^n$ (coming from multiplying $(-x^5)$ by $B_{n-5}x^{n-5}$). Therefore

$$A_n = B_n - B_{n-5}$$

or, rearranging things,

$$B_n = A_n + B_{n-5}.$$

By the same reasoning we can also find that

$$C_n = B_n + C_{n-10}$$
$$D_n = C_n + D_{n-25}$$
$$E_n = D_n + E_{n-50}.$$

For boundary conditions, we know that none of the series has any terms with x^n for $n < 0$, and so $A_n = B_n = C_n = D_n = E_n = 0$ for $n < 0$. We also know that $A_n = 1$ for all $n \geq 0$. Armed with this information, we can compute B_n for $n \geq 0$, making use of the recursion formula we've just worked out. Thus, $B_0 = A_0 + B_{-5} = 1 + 0 = 1$, $B_1 = A_1 + B_{-4} = 1 + 0 = 1$, . . . , $B_5 = A_5 + B_0 = 1 + 1 = 2$, etc. Once we've worked out some of the B's, we can start computing C's, and so on. Even so, working all the way out to E_{100} by hand could be time-consuming, though it wouldn't take long using a computer. But we can save a lot of effort by observing that we don't need *all* the intermediate numbers. To compute E_{100} we need to know E_{50}, and to compute that we need E_0. We also need D_{100}, D_{50}, and D_0. To compute *those*, we also need to know D_{75} and D_{25}, and so on. So if we plan ahead a little bit, we can compute only those elements we actually need.

Pólya demonstrated the process by beginning to fill in a table with $n = 0, 10, 20, \ldots, 100$. The second problem on the second homework assignment was to finish the computation and find E_{100}. (Pólya also provided as a hint that E_{50} happens to be 50.) He failed to point out that some intermediate multiples of 5 would also be necessary, but everyone seemed to figure that out anyway. The following table shows the minimum number of entries that need to be filled in to get the final answer of 292. (Some of the entries, such as B_{85} and B_{95}, could be left out by observing (and proving) some simple patterns, such as $B_n = A_n + B_{n-5} = A_n + (A_{n-5} + B_{n-10}) = 2 + B_{n-10}$ for $n \geq 10$, but we'll work them out anyway.)

n	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100
B_n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
C_n	1	2	4	6	9	12	16	20	25	30	36	42	49	56	64	72	81		100		121
D_n	1					13					49					121					242
E_n	1										50										292

Here is a summary of some of the more useful rules regarding generating functions. Suppose we have two generating functions: