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Yezid Donoso
Ramon Fabregat



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Dedication

To my wife, Adriana

For her love and tenderness and for our future together

To my children, Andres Felipe, Daniella, Marianna,

and the following with Adry

... a gift of God to my life

Yezid

To my wife, Telvys, and my children

... "continuarem caminant cap a Itaca"

Ramon

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Preface

Many new multicast applications emerging from the Internet, such as Voice-over-IP (VoIP), videoconference, TV over the Internet, radio over the Internet, video streaming multipoint, etc., have the following resource requirements: bandwidth consumption, end-to-end delay, delay jitter, packet loss ratio, and so forth. It is therefore necessary to formulate a proposal to specify and provide the resources necessary for these kinds of applications so they will function properly.

To show how these new applications can comply with these requirements, the book presents a multi-objective optimization scheme in which we will analyze and solve the problems related to resources optimization in computer networks. Once the readers have studied this book, they will be able to extend these models by adding new objective functions, new functions that act as restrictions, new network models, and new types of services or applications.

This book is for an academic and scientific setting. In the professional environment, it is focused on optimization of resources that a carrier needs to know to profit from computer resources and its network infrastructure. It is very useful as a textbook mainly for master's- or Ph.D.-level courses, whose subjects are related to computer networks traffic engineering, but it can also be used for an advanced or specialized course for the senior year of an undergraduate program. On the other hand, it can be of great use for a multi-objective optimization course that deals with graph theory by having represented the computer networks through graphs.

The book structure is as follows:

Chapter 1: Analyzes the basic optimization concepts, as well as several techniques and algorithms for the search of minimals.

Chapter 2: Analyzes the basic multi-objective optimization concepts and the ways to solve them through traditional techniques and several metaheuristics.

Chapter 3: Shows how to analytically model the computer network problems dealt with in this book.

Chapter 4: The book's main chapter — it shows the multi-objective models in computer networks and the applied way in which we can solve them.

Chapter 5: An extension of Chapter 4, applied to optical networks.

Chapter 6: An extension of Chapter 4, applied to wireless networks.

Lastly, Annex A provides the source code to solve the mathematical model problems presented in this book through solvers. Annex B includes some source codes programmed in C language, which solve some of the multi-objective optimization problems presented. These source files are available online at http://www.crcpress.com/e_products/downloads/default.asp

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He coordinated the participation of broadband communications and distributed systems research group (BCDS) in the ADAPTPlan project (a Spanish national research project). He is a member of the Spanish Network of Excellence in MPLS/GMPLS networks, which involves several Spanish institutions. He has participated in the technical program committees of several conferences and has coauthored several papers published in international journals and presented at leading international conferences.

Chapter 1

Optimization Concepts

In the field of engineering, solving a problem is not enough; the solution found must be the best solution possible. In other words, one must find the optimal solution to the problem. We say that this is the best possible solution because in the real world this problem may have certain constraints by which the solutions found may be feasible (they can be implemented in practice) and unfeasible (they cannot be implemented).

In engineering, one speaks of optimization when one wants to solve complex problems. Such complexity may be associated with the kind of problem one wants to solve (i.e., if the problem is nonlinear) or the kind of solution one wishes to get (i.e., whether the solution is exact or an approximation).

There are five basic ways to solve such problems: analytically, numerically, algorithmically through heuristics, algorithmically through metaheuristics, or through simulation. Analytical solutions are practically possible for simple problems, but complex or large-sized problems are very difficult and require too much computational time. When the analytical model is very complex, problems can be solved by approximation using numerical methods. To obtain such optimal approximate values, functions analyzed must usually meet a series of conditions. If such conditions are not met, the numerical method may converge toward the optimal value. In any event, these techniques are very useful when the problems are mono-objective, whether linear or not.

However, when a problem is multi-objective, numerical methods are susceptible of being nonconvergent, depending on the model used. For example, if one attempts to solve a multi-objective optimization scheme

by means of numerical methods using the mono-objective scheme of the weighted sum of the functions (which will be explained later), in addition to the conditions that the functions must meet for the specific numerical method, such a multi-objective scheme would present inconveniences if the search space is not convex, because one might not find many solutions. One can also find the solution applying computational algorithms called heuristics. In this case, heuristics presents a computational scheme that can reach the optimal value in an computational time. But the search of solutions with this type of algorithm may exhibit serious problems if, for example, the spaces are nonconvex or if the nature of the problem is of combined solution analysis. Furthermore, heuristics may present serious computational time problems when the problem is NP-Hard. A recognition problem P_1 is said to be NP-Hard if all other problems in the class NP polynomially to P_1 , and we say that a recognition problem P_1 is in the class NP if for every yes instance of P_1 , there is a short (i.e., polynomial length) verification that the instance is a yes instance [AHU93].

To overcome these inconveniences, metaheuristics have been created, which obtain an approximate solution to practically any kind of problem that is NP-Hard and complex or combined analysis solutions. Among existing metaheuristics we can mention genetic algorithms, Tabu search, ant colony, simulated annealing, memetic algorithms, etc. Many of these metaheuristics have been redesigned to provide solutions to multi-objective problems, which are the main interest of this book.

This chapter provides an introduction to fundamentals of local and global minimal and some existing techniques to search for such minimal.

1.1 Local Minimum

When optimizing a function $f(x)$, one wants to find the minimum value in an $[a, b]$ interval; that is, $a \leq x \leq b$. This minimum value is called the **local minimum** (Figure 1.1).

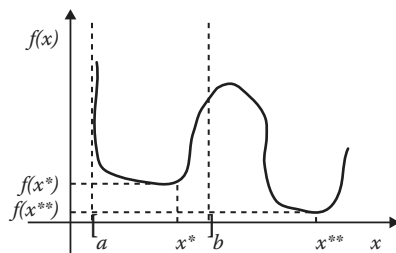


Figure 1.1 Local minimum.

If given function $f(x)$, we want to find the minimum value, but only in the $(a \leq x \leq b)$ interval. The resulting $f(x^*)$ value is called the local minimum of function $f(x)$ in interval $[a, b]$. As shown in Figure 1.1, this $f(x^*)$ value is the minimum value in the $[a, b]$ interval, but is not the minimum value of function $f(x)$ in the $(-\infty, \infty)$ interval.

Traditionally, search techniques for local minima are simpler than search techniques for global minima due, among many reasons, to the complexity generated in the search space when the interval is $(-\infty, \infty)$.

1.2 Global Minimum

When the function minimized is not constrained to a specific interval of the function, then one says that the value found is a **global minimum**. In this case, the search space interval is associated with $(-\infty, \infty)$.

Even though in Figure 1.1 the value of function $f(x^*)$ is a minimum, we can see that $f(x^{**}) < f(x^*)$. If there is no other value $f(x')$, so that $f(x') < f(x^{**})$ in the $(-\infty, \infty)$ interval, then one says that $f(x^{**})$ is a global minimum of function $f(x)$.

To find the **global maximum** value of a function, the analysis would be exactly the same, but in this case there should not exist another $f(x')$ value so that $f(x') > f(x^{**})$ in the $(-\infty, \infty)$ interval.

1.3 Convex and Nonconvex Sets

Definition 1

A set S of \mathfrak{R}^n is **convex** if for any pairs of points $P_1, P_2 \in S$ and for every $\lambda \in [0, 1]$ one proves that $P = \lambda P_1 + (1 - \lambda)P_2 \in S$. Point P is a linear combination of points P_1 and P_2 .

A set $S \subseteq \mathfrak{R}^n$ is convex if the linear combination of any two points in S also belongs to S .

On the other hand, a set is **nonconvex** if there is at least one point P in set S that cannot be represented by a linear combination.

Taking into account these definitions, Figure 1.2 represents convex solution sets and Figure 1.3 represents nonconvex solution sets.

Below are examples of convex sets.

Exercise

- a. Is set $S = \{(x_1, x_2) \in \mathfrak{R}^2 / x_2 \geq x_1\}$ convex?

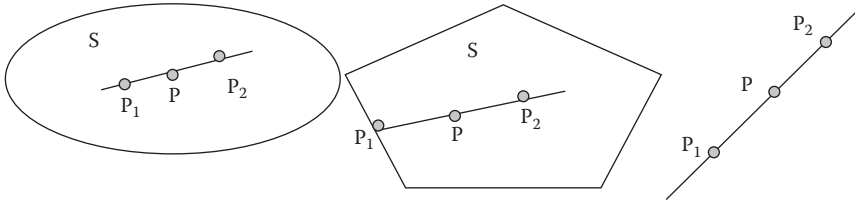


Figure 1.2 Convex sets.

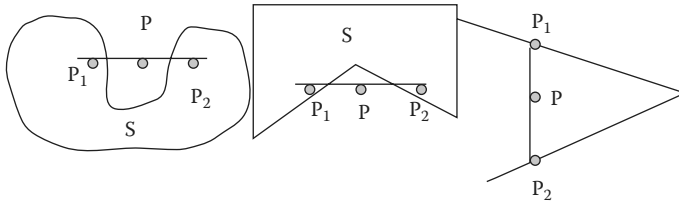


Figure 1.3 Nonconvex sets.

Proof:

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $x, y \in S$. We must prove that for every $\lambda \in [0, 1]$, $z = (z_1, z_2) = \lambda x + (1 - \lambda)y \in S$. In this case, we must prove that $z_2 \geq z_1$.

$$\lambda x + (1 - \lambda)y = (\lambda x_1 + (1 - \lambda)y_1, \lambda x_2 + (1 - \lambda)y_2)$$

Because $x, y \in S$, $x_2 \geq x_1$, $y_2 \geq y_1$, and $\lambda \geq 0$ $y (1 - \lambda) \geq 0$.

Then $\lambda x_2 \geq \lambda x_1$ $y (1 - \lambda)y_2 \geq (1 - \lambda)y_1$.

Adding both inequalities, we have

$$\lambda x_2 + (1 - \lambda)y_2 \geq \lambda x_1 + (1 - \lambda)y_1$$

Because $z \in S$, $z_2 = \lambda x_2 + (1 - \lambda)y_2$ $y z_1 = \lambda x_1 + (1 - \lambda)y_1$.

Replacing in the foregoing inequality we have that $z_2 \geq z_1$; therefore we have proven that S is a convex set.

- b. Let $S = \{x \in \mathfrak{R} / |x| \leq 1\}$. Is it convex?

Proof:

Let $x, y \in S$, that is, $|x| \leq 1$ and $|y| \leq 1$. We must prove that for every $\lambda \in [0, 1]$, $z = \lambda x + (1 - \lambda)y \in S$. In this case, we must prove that $|z| \leq 1$.

Because $z = \lambda x + (1 - \lambda)y$, applying $|\cdot|$ to the equality and taking into account the properties of this function — that $\lambda \in [0, 1]$, $|x| \leq 1$, and $|y| \leq 1$ — we have that $|z| = |\lambda x + (1 - \lambda)y| = |\lambda x| + |(1 - \lambda)y| = \lambda|x| + (1 - \lambda)|y| \leq 1$, and therefore S is convex.

1.4 Convex and Concave Functions

Definition 2

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined function of S in \mathfrak{R} . Function f is **convex** in S if and only if for any pair of points $P_1, P_2 \in S$ and for every $\lambda \in [0, 1]$ one proves that $f(\lambda P_1 + (1 - \lambda)P_2) \leq \lambda f(P_1) + (1 - \lambda)f(P_2)$.

Definition 3

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined function of S in \mathfrak{R} . Function f is **concave** in S if and only if for any pair of points $P_1, P_2 \in S$ and for every $\lambda \in [0, 1]$ one proves that $f(\lambda P_1 + (1 - \lambda)P_2) \geq \lambda f(P_1) + (1 - \lambda)f(P_2)$.

Definition 4

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined function of S in \mathfrak{R} . Function f is **strictly convex** in S if and only if for any pair of points $P_1, P_2 \in S$ and for every $\lambda \in [0, 1]$ one proves that $f(\lambda P_1 + (1 - \lambda)P_2) < \lambda f(P_1) + (1 - \lambda)f(P_2)$.

Definition 5

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined function of S in \mathfrak{R} . Function f is **strictly concave** in S if and only if for any pair of points $P_1, P_2 \in S$ and for every $\lambda \in [0, 1]$ one proves that $f(\lambda P_1 + (1 - \lambda)P_2) > \lambda f(P_1) + (1 - \lambda)f(P_2)$.

Exercise

Probe whether the following functions are concave or convex.

- a. Let $y = f(x) = ax + b$, in \mathfrak{R} , with $a, b \in \mathfrak{R}$.

Proof:

Let

$$\begin{aligned}
 f(\lambda x + (1 - \lambda)y) &= a[\lambda x + (1 - \lambda)y] + b \\
 &= a\lambda x + a(1 - \lambda)y + b \\
 &= \lambda ax + (1 - \lambda)ay + b + b - \lambda b \\
 &= \lambda ax + b + (1 - \lambda)ay + b - \lambda b \\
 &= \lambda[ax + b] + (1 - \lambda)ay + (1 - \lambda)b \\
 &= \lambda[ax + b] + (1 - \lambda)[ay + b] \\
 &= \lambda f(x) + (1 - \lambda)f(y)
 \end{aligned}$$

Consequently, function $y = f(x) = ax + b$ is convex and concave.

- b. Let $y = f(x) = \sum_{i=1}^n c_i x_i$, of \mathfrak{R}^n in \mathfrak{R} , with c_i, x_i positive.

Proof: Due to the function we are stating, in this case the inequality $<$ does not apply, because these types of functions are not convex.

Let

$$\begin{aligned}
 f(\lambda x + (1 - \lambda)y) &\geq \sum_{i=1}^n c_i (\lambda x_i + (1 - \lambda)y_i) \\
 &\geq \sum_{i=1}^n \lambda c_i x_i + \sum_{i=1}^n (1 - \lambda) c_i y_i
 \end{aligned}$$

$$\begin{aligned} &\geq \lambda \sum_{i=1}^n c_i x_i + (1 - \lambda) \sum_{i=1}^n c_i y_i \\ &\geq \lambda f(x) + (1 - \lambda) f(y) \end{aligned}$$

Consequently, function $y = f(x) = \sum_{i=1}^n c_i x_i$ is concave.

The following definitions can be used when function f is differentiable.

Definition 6

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined differential function of S in \mathfrak{R} . Then, function f is **convex** in S if and only if for any pair of points $P_1, P_2 \in S$ one proves that $[\nabla f(y) - \nabla f(x)](y - x) \geq 0$.

Definition 7

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined differential function of S in \mathfrak{R} . Then, function f is **strictly convex** in S if and only if for any pair of points $P_1, P_2 \in S$ one proves that $[\nabla f(y) - \nabla f(x)](y - x) > 0$.

Definition 8

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined differential function of S in \mathfrak{R} . Function f is **concave** in S if and only if for any pair of points $P_1, P_2 \in S$ one proves that $[\nabla f(y) - \nabla f(x)](y - x) \leq 0$.

Definition 9

Let S be a convex, unempty subset of \mathfrak{R}^n and f a defined differential function of S in \mathfrak{R} . Function f is **strictly concave** in S if and only if for any pair of points $P_1, P_2 \in S$ one proves that $[\nabla f(y) - \nabla f(x)](y - x) < 0$.

Exercise

Show whether the following functions are concave or convex.

- c. Let $y = f(x) = \sum_{i=1}^n x_i^2$, of \mathfrak{R}^n in \mathfrak{R} .

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