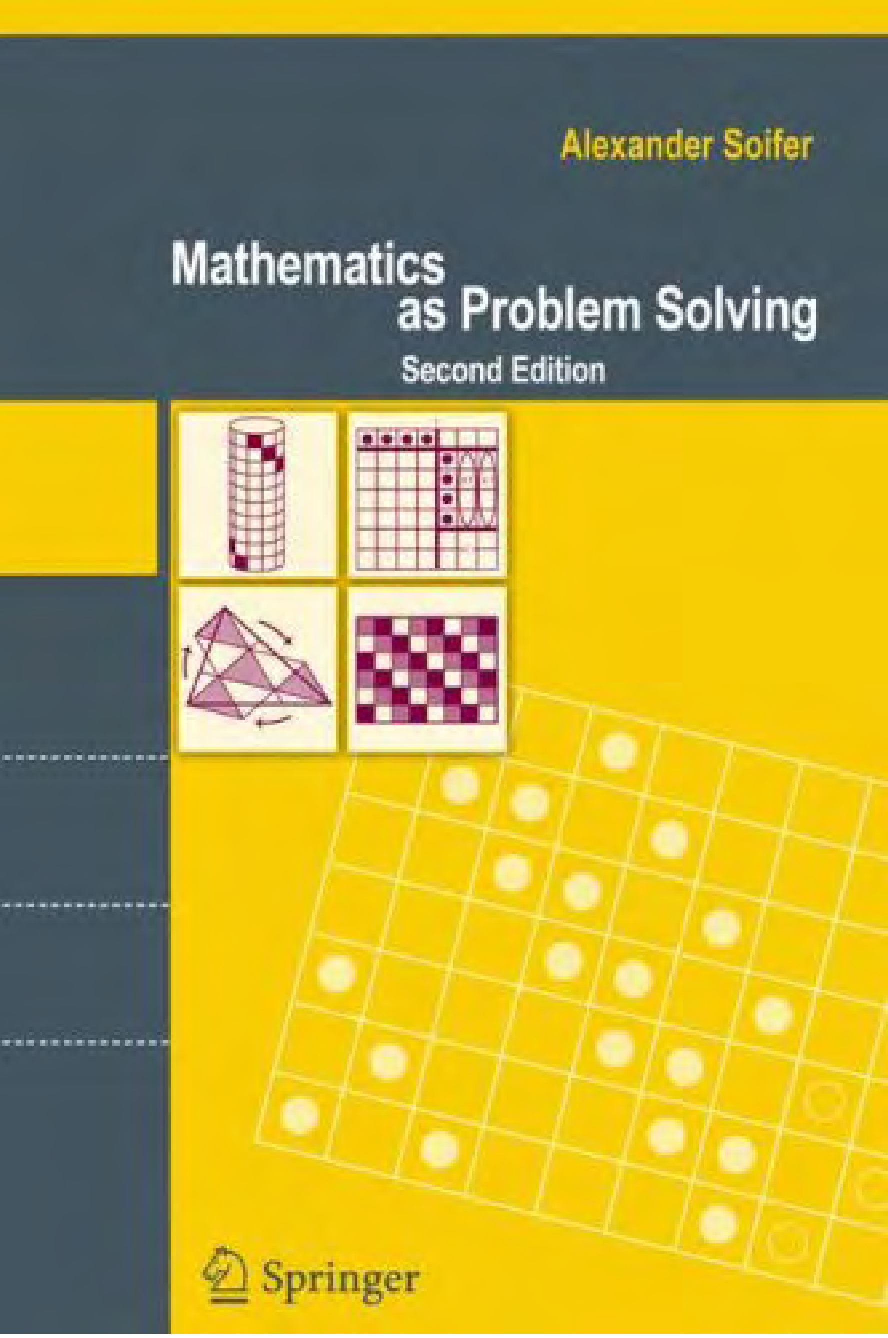
Alexander Soifer
Mathematics
as Problem Solving
Second Edition
Springer

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To Mark and Julia Soifer

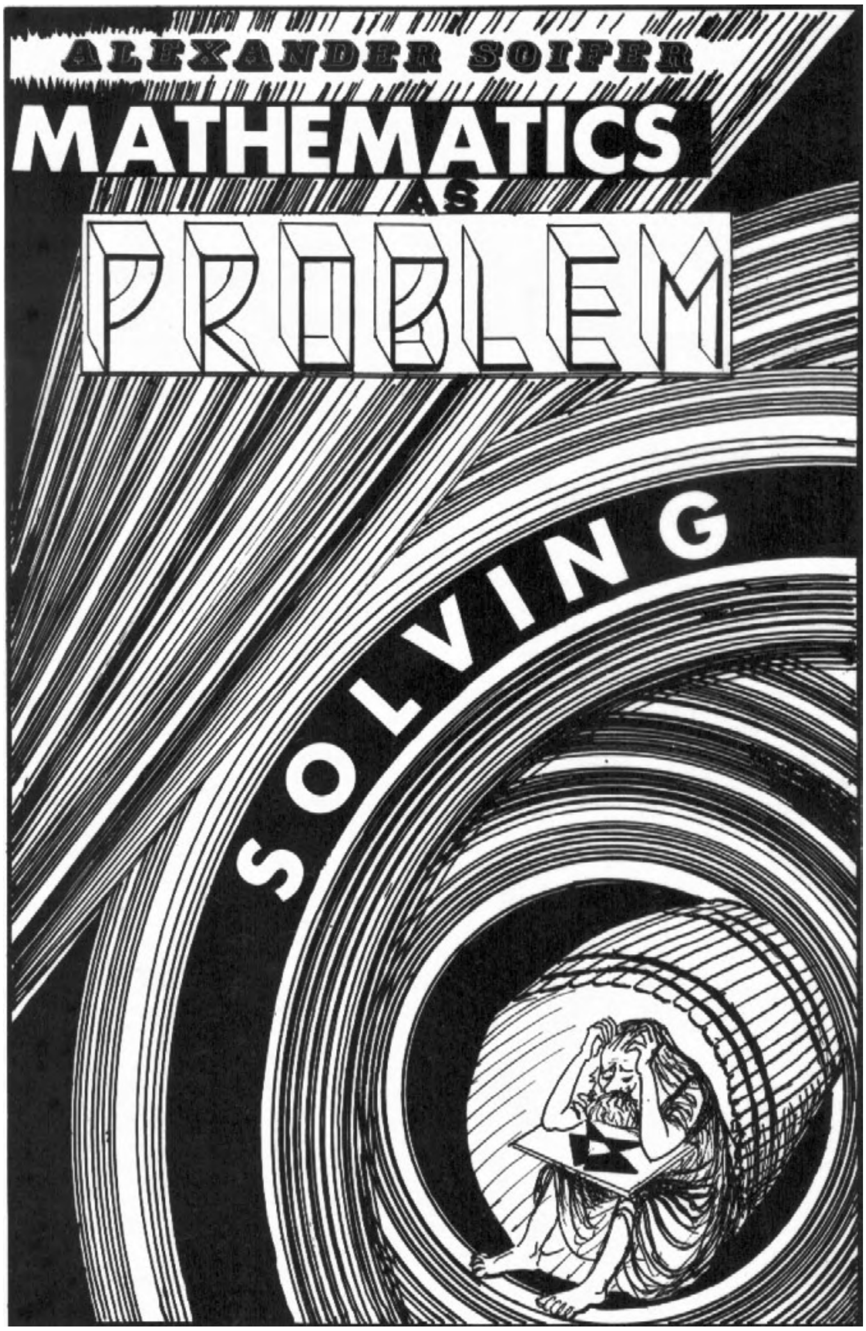
Frontispiece reproduces the front cover of the original edition. It was designed by my later father Yuri Soifer, who was a great artist. Will Robinson, who produced a documentary about him for the Colorado Springs affiliate of ABC, called him "an artist of the heart." For his first American one-man show at the University of Colorado in June–July 1981, Yuri sketched his autobiography:

I was born in 1907 in the little village Strizhevka in the Ukraine. From the age of three, I was taught at the Cheder (elementary school by a synagogue), and since that time I have been painting. At the age of ten, I entered Feinstein's Jewish High School in the city of Vinniza. The art teacher, Abram Markovich Cherkassky, a graduate of the Academy of Fine Arts at St. Petersburg, looked at my book of sketches of praying Jews, and consequently taught me for six years, until his departure for Kiev. Cherkassky was my first and most important teacher. He not only critiqued my work and explained various techniques, but used to sit down in my place and correct mistakes in my work until it was nearly unrecognizable. I couldn't then touch my work and continue – this was unforgettable.

In 1924, when I was 17, my relative, the American biologist, who later won the Nobel Prize in 1952, Selman A. Waksman, offered to take me to the United States to study and become an artist, and to introduce me to Chagall, but my mother did not allow this, and I went to Odessa to study at the Odessa Institute for the Fine Arts in the studio of Professor Mueller. Upon graduation in 1930, I worked at the Odessa State Jewish Theater, and a year later became the chief set and costume designer. In 1934, I came to Moscow to design plays for Birobidzhan Jewish Theater under the supervision of the great Michoels. I worked for the Jewish newspaper Der Emes*, the Moscow Film Studio, Theater of Lenin's Komsomol, and a permanent National Agricultural Exhibition. Upon finishing my 1941–1945 service in World War II, I worked for the National Exhibition in Moscow, VDNH.*

All my life, I have always worked in painting and graphics. Besides portraits and landscapes in oil, watercolor, gouache, and marker (and also acrylic upon the arrival in the USA), I was always inspired (perhaps, obsessed) by the images and ideas of the Russian Civil War, Word War II, biblical stories, and the little Jewish village that I came from.

The rest of my biography is in my works!



Front cover of the first edition, 1987, by Yuri Soifer.

Foreword

This book joins several other books available for the preparation of young scholars for a future that involves solving mathematical problems.

This training not only increases their fitness in competitions, but may also help them in other endeavors they may engage in the future.

The book is a diversified collection of problems from all areas of high school mathematics, and is written in a lively and engaging way.

The introductory explanations and worked problems help guide the reader without turning the additional problems into rote repetitions of the solved ones.

The book should become an essential tool in the armamentarium of faculty involved with training future competitors.

Branko Grünbaum
Professor of Mathematics
University of Washington
June 2008, Seattle, Washington

Foreword

This was the first of Alexander Soifer's books, I think, preceding *How Does One Cut a Triangle?* by a few years. It is short on anecdote and reminiscence, but there is charm in its youthful brusqueness and let's-get-right-to-business muscularity. And, mainly, there is a huge lode of problems, very good ones worked out and very good ones left to the reader to work out.

Every mathematician has his or her bag of tricks, and perhaps every mathematician will find some part of this book to view with smug condescension, but there may not be a mathematician alive that can so view all of this book. I notice that Paul Erdős registered his admiration for the chapters on combinatorics and geometry. For me, the Pigeonhole Principle problems were fascinating, exotic, and hard, and I would like to base a course on that section and on parts of the chapters on combinatorics and geometry.

Anyone coaching a Putnam Exam team should have a copy of this book, and anyone trying out for a Putnam Exam team would do well to train with this book. Training for prize exams is a good entree to higher mathematics, but even if you are not a competitive type, this book could well be the portal that will lead you into the wonderful world of mathematics.

Peter D. Johnson, Jr.
Professor of Mathematics
Auburn University
June 12, 2008, Auburn, Alabama

Foreword

In *Mathematics as Problem Solving*, Alexander Soifer has given an approach to problem solving that emphasizes basic techniques and thought rather than formulas. As he writes in the introduction to Chapter 2 (Numbers),

> *Numerous beautiful results could be presented here, but I will limit myself to problems illustrating some ideas and requiring practically no knowledge of number theory.*

The chapter headings are

- Language and a Few Celebrated Ideas
- Numbers
- Algebra
- Geometry
- Combinatorial Problems

Each topic is suitable for high school students, and there is a pleasant leanness to the list of topics (compare this with a current calculus text). The Chinese Remainder Theorem is out; the Pigeonhole Principle is in. As the reader will at some point discover, the Chinese Remainder Theorem can be deduced from the Pigeonhole Principle. Now is the time for fundamental problem solving; first things first. At the same time, nontrivial ruler and compass construction problems are basic to a proper understanding of geometry. Dr. Soifer has made a wise choice to emphasize this topic.

The 200 or so problems are well chosen to go with the emphasis on fundamental techniques, and they provide a rich resource. Some of the problems are appropriately routine, while some others are "little results" found by mathematicians in the course of their research. For example, Problem 1.29 is a rewording of a result mentioned in a survey paper by Paul Erdős; the discovery was originally made by Erdős and V.T. Sós. This problem also appeared on the 1979 USA Mathematical Olympiad.

> **1.29** (First Annual Southampton Mathematical Olympiad, 1986) An organization consisting of n members ($n > 5$) has $n+1$ three-member committees, no two of which have identical membership. Prove that there are two committees in which exactly one member is common.

Mathematics as Problem Solving is an ideal book with which to begin the study of problem solving. After readers have gone on to study more comprehensive sources, *Mathematics as Problem Solving* is likely to remain in a place of honor on their bookshelf.

Cecil Rousseau
Professor of Mathematics
Memphis State University
June 2008, Memphis, Tennessee

Preface to the Second Edition

The moving power of mathematical invention is not reasoning but imagination.

Augustus de Morgan

I released this book over twenty years ago. Since then she lived her own life, quite separately from me. Let me briefly trace her life here.

In March 1989, her title, *Mathematics as Problem Solving*, became the first "standard for school mathematics" of the National Council of Teachers of Mathematics [2]. In 1995, her French 4000-copy edition, *Les mathématiques par la résolution de problèmes, Éditions du Choix*, quickly sold out.

She was found charming and worthy by Paul Erdős, Martin Gardner, George Berszenyi, and others:

> *The problems faithfully reflect the world-famous Russian school of mathematics, whose folklore is carefully interwoven with more traditional topics. Many of the problems are drawn from the author's rich repertoire of personal experiences, dating back to his younger days as an outstanding competitor in his native Russia and spanning decades and continents as an organizer of competitions at the highest level.* – George Berzsenyi

> *The book contains a very nice collection of problems of various difficulties. I particularly liked the problems on combinatorics and geometry.* – Paul Erdős

> *Professor Soifer has put together a splendid collection of elementary problems designed to lead students into significant mathematical concepts and techniques. Highly recommended.* – Martin Gardner

In the "extended" *American Mathematical Monthly* review, Cecil Rousseau paid her a high compliment:

> *Retelling the best solutions and sharing the secrets of discovery are part of the process of teaching problem solving. Ideally, this process is characterized by mathematical skill, good taste, and wit. It is a characteristically personal process and the best such teachers have surely left their personal marks on students and readers. Alexander Soifer is a teacher of problem solving and his book,* Mathematics as Problem Solving, *is designed to introduce problem solving to the next generation.*

This poses a problem: how does one reach out to the next generation and charm it into reading and doing mathematics? I am deeply grateful to Ann Kostant for solving this problem by inviting a new edition of this book into the historic Springer. I thank Col. Dr. Robert Ewell for converting my sketches into real illustrations. I am so very grateful to the first readers of this manuscript, Branko Grünbaum, Peter D. Johnson, Jr., and Cecil Rousseau for their comments and forewords.

For the expanded Springer edition, I have added a sixth chapter dedicated to my favorite problem of the many problems that I have created, "Chess 7×7." I found three beautiful solutions to it. Moreover, this problem was inspired by the "serious" mathematics of Ramsey Theory, and once it was solved, it led me back to the "serious" mathematics of finite projective planes. I hope you will enjoy this additional chapter.

Let me mention for those who would like to read my other book that this book was followed by the books [9, 1, 10] listed in the bibliography. Then there came *The Mathematical Coloring Book* [11], after 18 years of writing. Books [12] and [13] will follow soon, as will new expanded editions of the books [9, 1, 10]. All my books will be published by Springer.

Write back to me; your solutions, problems, and ideas are always welcome!

Alexander Soifer
Colorado Springs, Colorado
May 8, 2008

Preface to the First Edition

Remember but him, who being demanded, to what purpose he toiled so much about an Art, which could by no means come to the knowledge of many. Few are enough for me; one will suffice, yea, less than one will content me, answered he. He said true: you and another are a sufficient theatre one for another; or you to your selfe alone!!

Michel de Montaigne
Of Solitarinesse. Essayes [6]

I was fortunate to grow up in the problem-solving atmosphere of Moscow with its mathematical clubs, schools, and Olympiads. The material for this book stems from my participation in numerous mathematical competitions of all levels, from school to national, as a competitor, an organizer, a judge, and a problem writer; but most importantly, from the mathematical folklore I grew up on.

This book contains about 200 problems, over one-third of which are discussed in detail, sometimes even with two or more solutions. When I started, I thought that beauty, challenge, elegance, and surprising results and solutions alone would determine my choices. During my work, however, one more factor powerfully forced itself into account: the interplay of selected problems.

This book is written for high school and college students, teachers, and everyone else desiring to experience the mystery and beauty of mathematics. It can be and has been used as a text for an undergraduate or graduate course or workshop on problem solving.

Auguste Renoir once said that just as some people all their lives read one book (the Bible, for example), so could he paint all his life one painting. I cannot agree with him more. This is the book I am going to write all my life. That is why I welcome so much your comments, corrections, ideas, alternative solutions, and suggestions to include other methods or to cover other areas of mathematics. Do send me your ideas and solutions: best of them as well as the names of their authors will be included in the future revised editions of this book. I hope, though, that this book will never reach the intimidating size of a calculus text.

One can fairly make an argument that this book is raw, unpolished. Perhaps that is not all bad: sketches by Modigliani give me, for one, so much more than sweated-out oils of Old Masters. Maybe a problem-solving book ought to be a sketch book!

To assign true authorship to these problems is as difficult as to folklore tales. The few references that I have given indicate my source rather than a definitive reference to the first mentioning of a problem. Even problems that I created and published myself might have existed before I was born!

I thank Valarie Barnes for bravely agreeing to type this manuscript; it was her first encounter of the mathematical kind. I thank my student Richard Jessop for producing such a masterpiece of typesetting art.

I am grateful to my parents Yuri and Rebbeca for introducing me to the world of arts, and to my children Mark and Julia for inspiration. My special thanks go to the first judges of this manuscript, my students in Colorado Springs and Southampton for their enthusiasm, ideas, and support.

A. Soifer
Colorado Springs, Colorado
November 1986

Contents

1

Language and Some Celebrated Ideas

1.1 Streetcar Stories

I would like to start our discussion with the following stories.

Streetcar Story I

You enter a streetcar with six other passengers on the first stop of its route. On the second stop, four people come in and two get off. On the third stop, seven people come in and five get off. On the fourth stop, eight people come in and three get off. On the fifth stop, thirteen people come in and eight get off.

How old is the driver?

Did you start counting passengers in the streetcar? If you did, here is your first lesson: *Do not start solving a problem before you read it!*

Sounds obvious? Perhaps you are right. But you should not underestimate its importance. I for one underestimated some obvious things in life, and had to learn the hard way lessons like, “Do not read while you drive!”

The story above does not give us any information relevant to the age of the driver. However, relevance of information is not always obvious.

Streetcar Story II

The reunion of two friends in a streetcar sounded like this:

A. Soifer, *Mathematics as Problem Solving*, DOI: 10.1007/978-0-387-74647-0_1,

— How are you? Thank you, I am fine.

— You just got married when we met last. Any children?

— I have three kids!

— Wow! How old are they?

— Well, if you multiply their ages, you would get 36; but if you add them up, you'd get the number of passengers in this streetcar.

— Gotcha, but you did not tell me enough to figure out their ages.

— My oldest kid is a great sportsman.

— Aha! Now I know their ages!

Find the number of passengers in the streetcar and the ages of the children.

Can the statement "my oldest kid is a great sportsman" have any relevance? It can. In fact, it does! Moreover, the fact that without this statement the second friend cannot figure out the ages of the children carries valuable information, too!

Let us take a look at the following table:

Decompositions of 36 into 3 factors x, y, z	The sum $x + y + z$
$1 \cdot 1 \cdot 36$	38
$1 \cdot 2 \cdot 18$	21
$1 \cdot 3 \cdot 12$	16
$1 \cdot 4 \cdot 9$	14
$1 \cdot 6 \cdot 6$	13
$2 \cdot 2 \cdot 9$	13
$2 \cdot 3 \cdot 6$	11
$3 \cdot 3 \cdot 4$	10

The fact that the second friend was unable to figure out the ages x, y, z of the children when he knew their sum $x + y + z$ implies that there must be at least two solutions for the given sum $x+y+z$ of ages! The table shows that only 13 appears twice in the column $x + y + z$; therefore, $x+y+z = 13$, and we know the number of passengers! We can also see the relevance of the oldest kid being a great sportsman: it rules out 1, 6, 6 and leaves 2, 2, 9!

1.2 Language

As with any other science, mathematics is formulated in an ordinary language — English in the United States. It is essential to use language correctly as well as to correctly interpret sentences. I have no intention to discuss formal mathematical language. I would like, however, to briefly talk about constructing complex sentences, and to define the meaning of "not", "and", "or", "imply," "if and only if", etc.

We will deal only with statements that are clearly true or false in a given context.

Here are a few examples of such statements:

(1) Chicago is the capital of the United States.
(2) One yard is equal to three feet.
(3) Any sports car is red.
(4) Any Ferrari is red.

As you can see, the first and third statements are false and the second statement is true. It took me a visit to my friend Bob Penkhus, a car dealer, to find out that the fourth statement is false.

The truth or falsity of a composite statement is completely determined by the truth or falsity of its components.

Negation

Given a statement A. The negation of A, denoted by $\neg A$ and read "not A," is a new statement, which is understood to assert that "A is false."

Let 1 stand for true and 0 stand for false. Then the following table defines the values of $\neg A$:

A	$\neg A$
1	0
0	1

i.e., $\neg A$ is false when A is true, and $\neg A$ is true when A is false.

Conjunction

Given statements A and B. The conjunction of A and B, denoted $A \wedge B$ and read "A and B," is a new statement which is understood to assert that "A is true and B is true." The following truth table defines $A \wedge B$:

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction

Given statements A and B. The disjunction of A and B, denoted $A \vee B$, and read "A or B," is a new statement that is understood to assert "at least one of the statements A, B is true." The following truth table defines $A \vee B$:

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

Implication

Given statements A and B. The implication $A \Rightarrow B$, to be read "A implies B," is a statement that is understood to assert that "if A is true, then B is true." It is defined by the following truth table:

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

Please note that the meaning of "implication" in mathematics is quite different from the common language use of this word: $A \Rightarrow B$ is false only if A is true and B is false.

Equivalence

Given statements A and B. The equivalence $A \Leftrightarrow B$, to be read "A equivalent B," is an abbreviation for the following statement:

$$(A \Rightarrow B) \wedge (B \Rightarrow A).$$

In order to uniquely interpret a composite statement, we sometimes need to use lots of parentheses. This can make a statement quite difficult to read or evaluate. We can resolve this problem exactly the same way we do in arithmetic: by establishing the order of operations in a parenthesis-free composite statement:

$$\begin{array}{rcl} \text{We apply} & \neg & \text{first} \\ & \wedge & \text{second} \\ & \vee & \text{third} \\ & \Rightarrow & \text{fourth} \\ & \Leftrightarrow & \text{fifth.} \end{array}$$

Finally, a composite statement that is true regardless of the truth or falsity of its components is called a *tautology*.

Problems

Prove the following tautologies:

1.1. $A \Rightarrow A$

1.2. $A \Rightarrow A \vee B$

1.3. $A \wedge B \Rightarrow A$

1.4. $\neg\neg A \Leftrightarrow A$

1.5. $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$ (De Morgan's Law)

1.6. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$ (De Morgan's Law)

1.7. $(A \Rightarrow B) \wedge (B \Rightarrow C) \Rightarrow (A \Rightarrow C)$

1.8. $(\neg B \Rightarrow \neg A) \Leftrightarrow (A \Rightarrow B)$

1.9. $(A \wedge \neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$

1.10. $(A \wedge \neg B \Rightarrow \mathcal{F}) \Rightarrow (A \Rightarrow B)$, ($\mathcal{F}$ denotes a false statement)

From now on we will use symbols:

$\wedge$ for "and"
$\vee$ for "or"
$\Rightarrow$ for "implies"
$\Leftrightarrow$ for "if and only if"
$\neg$ for "not"
$\exists$ for "there exists"
$\forall$ for "for every".

If $A \Rightarrow B$ is true, we say that A is a sufficient condition for B; at the same time, we say that B is a necessary condition for A. If $A \Leftrightarrow B$ is true, then B is said to be a necessary and sufficient condition for A. Please remember that a statement *converse* to $A \Rightarrow B$ is $B \Rightarrow A$. A statement *opposite* to $A \Rightarrow B$ is $\neg(A \Rightarrow B)$.

1.3 Arguing by Contradiction

Problems 1.9 and 1.10 presented the following tautologies:

$$(A \wedge \neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)$$
$$(A \wedge \neg B \Rightarrow \mathcal{F}) \Rightarrow (A \Rightarrow B).$$

These two tautologies justify a celebrated method of mathematical proof: arguing by contradiction.

Let us say we are given that A is true and we are asked to prove that B is true. We assume that B is not true — i.e., $\neg B$ is true — and then start with A and $\neg B$ and continue deducing until we arrive at a contradiction to what is given — i.e., at $\neg A$ — what is known to be true.

1.11. Prove the sum of a rational number and an irrational number is an irrational number.

Proof. Let r be a rational number (i.e., $r = m/n$ for some integers m, n with $n \neq 0$), and let i be an irrational number (i.e., i cannot be presented in the form s/t, where s, t are integers and $t \neq 0$).

Assume that the sum $r+i$ is a rational number, say r_1. Then if $r_1 = p/q$ for integers p, q, with $q \neq 0$, we get

$$\begin{aligned} r + i &= r_1 \\ i &= r_1 - r \\ i &= \frac{p}{q} - \frac{m}{n} \\ i &= \frac{np - mq}{nq}. \end{aligned}$$

That is, i is a rational number, which contradicts the given fact that i is an irrational number. Therefore, $r+i$ is irrational. □

1.12. (*Pigeonhole Principle*) If $kn+1$ pigeons (k, n are positive integers) are placed in n pigeonholes, then at least one of the holes contains at least $k+1$ pigeons.

Proof. Assume that there are no holes that contain at least $k+1$ pigeons. Then:

$$\begin{array}{rl} \text{the 1st hole contains} & \leq k \text{ pigeons} \\ \text{the 2nd hole contains} & \leq k \text{ pigeons} \\ \vdots & \\ + \ \text{the } n\text{th hole contains} & \leq k \text{ pigeons} \\ \hline \text{total number of pigeons} & \leq k \times n \text{ pigeons} \end{array}$$

This contradicts the given fact that there are $kn+1$ pigeons. Therefore, there is a hole that contains at least $k+1$ pigeons. □

Problems

1.13. Prove that the product of a nonzero rational number and an irrational number is again an irrational number.

As you probably know, a positive integer greater than 1 is called *prime* if it has exactly two divisors, 1 and itself. The Fundamental Theorem of Arithmetic states that any positive integer greater than 1 can be decomposed into the product of prime numbers and that this decomposition is unique up to the order of factors.