

A REFERENCE BOOK
FOR THE MECHANICAL ENGINEER, DESIGNER,
MANUFACTURING ENGINEER, DRAFTSMAN,
TOOLMAKER, AND MACHINIST

27th Edition
**Machinery's
Handbook**

BY ERIK OBERG, FRANKLIN D. JONES,
HOLBROOK L. HORTON, AND HENRY H. RYFFEL

CHRISTOPHER J. MCCAULEY, EDITOR
RICCARDO M. HEALD, ASSOCIATE EDITOR
MUHAMMED IQBAL HUSSAIN, ASSOCIATE EDITOR

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PREFACE

Machinery's Handbook has served as the principal reference work in metalworking, design and manufacturing facilities, and in technical schools and colleges throughout the world, for more than 90 years of continuous publication. Throughout this period, the intention of the *Handbook* editors has always been to create a comprehensive and practical tool, combining the most basic and essential aspects of sophisticated manufacturing practice. A tool to be used in much the same way that other tools are used, to make and repair products of high quality, at the lowest cost, and in the shortest time possible.

The essential basics, material that is of proven and everlasting worth, must always be included if the *Handbook* is to continue to provide for the needs of the manufacturing community. But, it remains a difficult task to select suitable material from the almost unlimited supply of data pertaining to the manufacturing and mechanical engineering fields, and to provide for the needs of design and production departments in all sizes of manufacturing plants and workshops, as well as those of job shops, the hobbyist, and students of trade and technical schools.

The editors rely to a great extent on conversations and written communications with users of the *Handbook* for guidance on topics to be introduced, revised, lengthened, shortened, or omitted. In response to such suggestions, in recent years material on logarithms, trigonometry, and sine-bar constants have been restored after numerous requests for these topics. Also at the request of users, in 1997 the first ever large-print or "desktop" edition of the *Handbook* was published, followed in 1998 by the publication of *Machinery's Handbook CD-ROM* including hundreds of additional pages of material restored from earlier editions. The large-print and CD-ROM editions have since become permanent additions to the growing family of *Machinery's Handbook* products.

Regular users of the *Handbook* will quickly discover some of the many changes embodied in the present edition. One is the combined *Mechanics and Strength of Materials* section, arising out of the two former sections of similar name; another is the *Index of Standards*, intended to assist in locating standards information. "Old style" numerals, in continuous use in the first through twenty-fifth editions, are now used only in the index for page references, and in cross reference throughout the text. The entire text of this edition, including all the tables and equations, has been reset, and a great many of the numerous figures have been redrawn. This edition contains more information than ever before, and sixty-four additional pages brings the total length of the book to 2704 pages, the longest *Handbook* ever.

The 27th edition of the *Handbook* contains significant format changes and major revisions of existing content, as well as new material on a variety of topics. The detailed tables of contents located at the beginning of each section have been expanded and fine tuned to simplify locating your topic; numerous major sections have been extensively reworked and renovated throughout, including *Mathematics*, *Mechanics and Strength of Materials*, *Properties of Materials*, *Fasteners*, *Threads and Threading*, and *Unit Conversions*. New material includes fundamentals of basic math operations, engineering economic analysis, matrix operations, disc springs, constants for metric sine-bars, additional screw thread data and information on obscure and historical threads, aerodynamic lubrication, high speed machining, grinding feeds and speeds, machining econometrics, metalworking fluids, ISO surface texture, pipe welding, geometric dimensioning and tolerancing, gearing, and EDM.

Other subjects in the *Handbook* that are new or have been revised, expanded, or updated are: analytical geometry, formulas for circular segments, construction of four-arc ellipse, geometry of rollers on a shaft, mechanisms, additional constants for measuring weight of piles, Ohm's law, binary multiples, force on inclined planes, and measurement over pins.

The large-print edition is identical to the traditional toolbox edition, but the size is increased by a comfortable 140% for easier reading, making it ideal as a desktop reference. Other than size, there are no differences between the toolbox and large-print editions.

PREFACE

The *Machinery's Handbook 27* CD-ROM contains the complete contents of the printed edition, presented in Adobe Acrobat PDF format. This popular and well known format enables viewing and printing of pages, identical to those of the printed book, rapid searching, and the ability to magnify the view of any page. Navigation aids in the form of thousands of clickable bookmarks, page cross references, and index entries take you instantly to any page referenced.

The CD contains additional material that is not included in the toolbox or large print editions, including an extensive index of materials referenced in the Handbook, numerous useful mathematical tables, sine-bar constants for sine-bars of various lengths, material on cement and concrete, adhesives and sealants, recipes for coloring and etching metals, forge shop equipment, silent chain, worm gearing and other material on gears, and other topics.

Also new on the CD are numerous interactive math problems. Solutions are accessed from the CD by clicking an icon, located in the page margin adjacent to a covered problem, (see figure shown here). An internet connection is required to use these problems. The list of interactive math solutions currently available can be found in the *Index of Interactive Equations*, starting on page 2689. Additional interactive solutions will be added from time to time as the need becomes clear.

Those users involved in aspects of machining and grinding will be interested in the topics Machining Econometrics and Grinding Feeds and Speeds, presented in the *Machining* section. The core of all manufacturing methods start with the cutting edge and the metal removal process. Improving the control of the machining process is a major component necessary to achieve a **Lean chain** of manufacturing events. These sections describe the means that are necessary to get metal cutting processes under control and how to properly evaluate the decision making.

A major goal of the editors is to make the *Handbook* easier to use. The 27th edition of the *Handbook* continues to incorporate the timesaving thumb tabs, much requested by users in the past. The table of contents pages beginning each major section, first introduced for the 25th edition, have proven very useful to readers. Consequently, the number of contents pages has been increased to several pages each for many of the larger sections, to more thoroughly reflect the contents of these sections. In the present edition, the *Plastics* section, formerly a separate thumb tab, has been incorporated into the *Properties of Materials* section. A major task in assembling this edition has been the expansion and reorganization of the index. For the first time, most of the many Standards referenced in the *Handbook* are now included in a separate *Index Of Standards* starting on page 2677.

The editors are greatly indebted to readers who call attention to possible errors and defects in the *Handbook*, who offer suggestions concerning the omission of some matter that is considered to be of general value, or who have technical questions concerning the solution of difficult or troublesome *Handbook* problems. Such dialog is often invaluable and helps to identify topics that require additional clarification or are the source of reader confusion. Queries involving *Handbook* material usually entail an in depth review of the topic in question, and may result in the addition of new material to the *Handbook* intended to resolve or clarify the issue. The new material on the mass moment of inertia of hollow circular rings, page 248, and on the effect of temperature on the radius of thin circular rings, page 405, are good examples.

Our goal is to increase the usefulness of the *Handbook* to the greatest extent possible. All criticisms and suggestions about revisions, omissions, or inclusion of new material, and requests for assistance with manufacturing problems encountered in the shop are always welcome.

Christopher J. McCauley, Senior Editor

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The editors would like to acknowledge all those who contributed ideas, suggestions, and criticisms concerning the *Handbook*.

Most importantly, we thank the readers who have contacted us with suggestions for new topics to present in this edition of the *Handbook*. We are grateful for your continuing constructive suggestions and criticisms with regard to *Handbook* topics and presentation. Your comments for this edition, as well as past and future ones are invaluable, and well appreciated.

Special thanks are also extended to current and former members of our staff, the talented engineers, recent-graduates, who performed much of the fact checking, calculations, artwork, and standards verification involved in preparing the printed and CD-ROM editions of the *Handbook*.

Many thanks to Janet Romano for her great *Handbook* cover designs. Her printing, packaging, and production expertise are irreplaceable, continuing the long tradition of *Handbook* quality and ruggedness.

Many of the American National Standards Institute (ANSI) Standards that deal with mechanical engineering, extracts from which are included in the *Handbook*, are published by the American Society of Mechanical Engineers (ASME), and we are grateful for their permission to quote extracts and to update the information contained in the standards, based on the revisions regularly carried out by the ASME.

ANSI Standards are copyrighted by the publisher. Information regarding current editions of any of these Standards can be obtained from ASME International, Three Park Avenue, New York, NY 10016, or by contacting the American National Standards Institute, West 42nd Street, New York, NY 10017, from whom current copies may be purchased. Additional information concerning Standards nomenclature and other Standards bodies that may be of interest is located on page 2079.

Several individuals in particular, contributed substantial amounts of time and information to this edition.

Mr. David Belforte, for his thorough contribution on lasers.

Manfred K. Brueckner, for his excellent presentation of formulas for circular segments, and for the material on construction of the four-arc oval.

Dr. Bertil Colding, provided extensive material on grinding speeds, feeds, depths of cut, and tool life for a wide range of materials. He also provided practical information on machining econometrics, including tool wear and tool life and machining cost relationships.

Mr. Edward Craig contributed information on welding.

Dr. Edmund Isakov, contributed material on coned disc springs as well as numerous other suggestions related to hardness scales, material properties, and other topics.

Mr. Sidney Kravitz, a frequent contributor, provided additional data on weight of piles, excellent proof reading assistance, and many useful comments and suggestions concerning many topics throughout the book.

Mr. Richard Kuzmack, for his contributions on the subject of dividing heads, and additions to the tables of dividing head indexing movements.

Mr. Robert E. Green, as editor emeritus, contributed much useful, well organized material to this edition. He also provided invaluable practical guidance to the editorial staff during the *Handbook's* compilation.

Finally, Industrial Press is extremely fortunate that Mr. Henry H. Ryffel, author and editor of *Machinery's Handbook*, continues to be deeply involved with the *Handbook*. Henry's ideas, suggestions, and vision are deeply appreciated by everyone who worked on this book.

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NUMBERS, FRACTIONS, AND DECIMALS

Table 1. Fractional and Decimal Inch to Millimeter, Exact^a Values

Fractional Inch	Decimal Inch	Millimeters	Fractional Inch	Decimal Inch	Millimeters
1/64	0.015625	0.396875		0.511811024	13
1/32	0.03125	0.79375	33/64	0.515625	13.096875
	0.039370079	1	17/32	0.53125	13.49375
3/64	0.046875	1.190625	35/64	0.546875	13.890625
1/16	0.0625	1.5875		0.551181102	14
5/64	0.078125	1.984375	9/16	0.5625	14.2875
	0.078740157	2	37/64	0.578125	14.684375
1/12	0.0833 ^b	2.1166	7/12	0.5833	14.8166
3/32	0.09375	2.38125		0.590551181	15
7/64	0.109375	2.778125	19/32	0.59375	15.08125
	0.118110236	3	39/64	0.609375	15.478125
1/8	0.125	3.175	5/8	0.625	15.875
9/64	0.140625	3.571875		0.62992126	16
5/32	0.15625	3.96875	41/64	0.640625	16.271875
	0.157480315	4	21/32	0.65625	16.66875
1/6	0.166	4.233	2/3	0.66	16.933
11/64	0.171875	4.365625		0.669291339	17
3/16	0.1875	4.7625	43/64	0.671875	17.065625
	0.196850394	5	11/16	0.6875	17.4625
13/64	0.203125	5.159375	45/64	0.703125	17.859375
7/32	0.21875	5.55625		0.708661417	18
15/64	0.234375	5.953125	23/32	0.71875	18.25625
	0.236220472	6	47/64	0.734375	18.653125
1/4	0.25	6.35		0.748031496	19
17/64	0.265625	6.746875	3/4	0.75	19.05
	0.275590551	7	49/64	0.765625	19.446875
9/32	0.28125	7.14375	25/32	0.78125	19.84375
19/64	0.296875	7.540625		0.787401575	20
5/16	0.3125	7.9375	51/64	0.796875	20.240625
	0.31496063	8	13/16	0.8125	20.6375
21/64	0.328125	8.334375		0.826771654	21
1/3	0.33	8.466	53/64	0.828125	21.034375
11/32	0.34375	8.73125	27/32	0.84375	21.43125
	0.354330709	9	55/64	0.859375	21.828125
23/64	0.359375	9.128125		0.866141732	22
3/8	0.375	9.525	7/8	0.875	22.225
25/64	0.390625	9.921875	57/64	0.890625	22.621875
	0.393700787	10		0.905511811	23
13/32	0.40625	10.31875	29/32	0.90625	23.01875
5/12	0.4166	10.5833	11/12	0.9166	23.2833
27/64	0.421875	10.715625	59/64	0.921875	23.415625
	0.433070866	11	15/16	0.9375	23.8125
7/16	0.4375	11.1125		0.94488189	24
29/64	0.453125	11.509375	61/64	0.953125	24.209375
15/32	0.46875	11.90625	31/32	0.96875	24.60625
	0.472440945	12		0.984251969	25
31/64	0.484375	12.303125	63/64	0.984375	25.003125
1/2	0.5	12.7			

^aTable data are based on 1 inch = 25.4 mm, exactly. Inch to millimeter conversion values are exact. Whole number millimeter to inch conversions are rounded to 9 decimal places.

^bNumbers with an overbar, repeat indefinitely after the last figure, for example $0.08\overline{33} = 0.08333\dots$

Numbers

Numbers are the basic instrumentation of computation. Calculations are made by operations of numbers. The whole numbers greater than zero are called natural numbers. The first ten numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are called numerals. Numbers follow certain formulas. The following properties hold true:

Associative law: $x + (y + z) = (x + y) + z$, $x(yz) = (xy)z$

Distributive law: $x(y + z) = xy + xz$

Commutative law: $x + y = y + x$

Identity law: $0 + x = x$, $1x = x$

Inverse law: $x - x = 0$, $x/x = 1$

Positive and Negative Numbers.—The degrees on a thermometer scale extending upward from the zero point may be called *positive* and may be preceded by a plus sign; thus +5 degrees means 5 degrees above zero. The degrees below zero may be called *negative* and may be preceded by a minus sign; thus, -5 degrees means 5 degrees below zero. In the same way, the ordinary numbers 1, 2, 3, etc., which are larger than 0, are called positive numbers; but numbers can be conceived of as extending in the other direction from 0, numbers that, in fact, are less than 0, and these are called negative. As these numbers must be expressed by the same figures as the positive numbers they are designated by a minus sign placed before them, thus: (-3). A negative number should always be enclosed within parentheses whenever it is written in line with other numbers; for example: $17 + (-13) - 3 \times (-0.76)$.

Negative numbers are most commonly met with in the use of logarithms and natural trigonometric functions. The following rules govern calculations with negative numbers.

A negative number can be added to a positive number by subtracting its numerical value from the positive number.

Example: $4 + (-3) = 4 - 3 = 1$

A negative number can be subtracted from a positive number by adding its numerical value to the positive number.

Example: $4 - (-3) = 4 + 3 = 7$

A negative number can be added to a negative number by adding the numerical values and making the sum negative.

Example: $(-4) + (-3) = -7$

A negative number can be subtracted from a larger negative number by subtracting the numerical values and making the difference negative.

Example: $(-4) - (-3) = -1$

A negative number can be subtracted from a smaller negative number by subtracting the numerical values and making the difference positive.

Example: $(-3) - (-4) = 1$

If in a subtraction the number to be subtracted is larger than the number from which it is to be subtracted, the calculation can be carried out by subtracting the smaller number from the larger, and indicating that the remainder is negative.

Example: $3 - 5 = -(5 - 3) = -2$

When a positive number is to be multiplied or divided by a negative number, multiply or divide the numerical values as usual; the product or quotient, respectively, is negative. The same rule is true if a negative number is multiplied or divided by a positive number.

Examples: $4 \times (-3) = -12$ $(-4) \times 3 = -12$

$15 \div (-3) = -5$ $(-15) \div 3 = -5$

When two negative numbers are to be multiplied by each other, the product is positive. When a negative number is divided by a negative number, the quotient is positive.

Examples: $(-4) \times (-3) = 12$; $(-4) \div (-3) = 1.333$

The two last rules are often expressed for memorizing as follows: "Equal signs make plus, unequal signs make minus."

Sequence of Performing Arithmetic Operations.—When several numbers or quantities in a formula are connected by signs indicating that additions, subtractions, multiplications, and divisions are to be made, the multiplications and divisions should be carried out first, in the sequence in which they appear, before the additions or subtractions are performed.

Example: $10 + 26 \times 7 - 2 = 10 + 182 - 2 = 190$
 $18 \div 6 + 15 \times 3 = 3 + 45 = 48$
 $12 + 14 \div 2 - 4 = 12 + 7 - 4 = 15$

When it is required that certain additions and subtractions should precede multiplications and divisions, use is made of parentheses () and brackets []. These signs indicate that the calculation inside the parentheses or brackets should be carried out completely by itself before the remaining calculations are commenced. If one bracket is placed inside another, the one inside is first calculated.

Example: $(6 - 2) \times 5 + 8 = 4 \times 5 + 8 = 20 + 8 = 28$
 $6 \times (4 + 7) \div 22 = 6 \times 11 \div 22 = 66 \div 22 = 3$
 $2 + [10 \times 6(8 + 2) - 4] \times 2 = 2 + [10 \times 6 \times 10 - 4] \times 2$
 $= 2 + [600 - 4] \times 2 = 2 + 596 \times 2 = 2 + 1192 = 1194$

The parentheses are considered as a sign of multiplication; for example:

$$6(8 + 2) = 6 \times (8 + 2).$$

The line or bar between the numerator and denominator in a fractional expression is to be considered as a division sign. For example,

$$\frac{12 + 16 + 22}{10} = (12 + 16 + 22) \div 10 = 50 \div 10 = 5$$

In formulas, the multiplication sign (\times) is often left out between symbols or letters, the values of which are to be multiplied. Thus,

$$AB = A \times B \quad \text{and} \quad \frac{ABC}{D} = (A \times B \times C) \div D$$

Ratio and Proportion.—The *ratio* between two quantities is the quotient obtained by dividing the first quantity by the second. For example, the ratio between 3 and 12 is $\frac{1}{4}$, and the ratio between 12 and 3 is 4. Ratio is generally indicated by the sign (:); thus, 12 : 3 indicates the ratio of 12 to 3.

A *reciprocal*, or *inverse* ratio, is the opposite of the original ratio. Thus, the inverse ratio of 5 : 7 is 7 : 5.

In a *compound* ratio, each term is the product of the corresponding terms in two or more simple ratios. Thus, when

$$8:2 = 4 \quad 9:3 = 3 \quad 10:5 = 2$$

then the compound ratio is

$$8 \times 9 \times 10 : 2 \times 3 \times 5 = 4 \times 3 \times 2$$

$$720:30 = 24$$

Proportion is the equality of ratios. Thus,

$$6:3 = 10:5 \quad \text{or} \quad 6:3::10:5$$

The first and last terms in a proportion are called the *extremes*; the second and third, the *means*. The product of the extremes is equal to the product of the means. Thus,

$$25:2 = 100:8 \quad \text{and} \quad 25 \times 8 = 2 \times 100$$

If three terms in a proportion are known, the remaining term may be found by the following rules:

The first term is equal to the product of the second and third terms, divided by the fourth.

The second term is equal to the product of the first and fourth terms, divided by the third.

The third term is equal to the product of the first and fourth terms, divided by the second.

The fourth term is equal to the product of the second and third terms, divided by the first.

Example: Let x be the term to be found, then,

$$x : 12 = 3.5 : 21 \quad x = \frac{12 \times 3.5}{21} = \frac{42}{21} = 2$$

$$\frac{1}{4} : x = 14 : 42 \quad x = \frac{\frac{1}{4} \times 42}{14} = \frac{1}{4} \times 3 = \frac{3}{4}$$

$$5 : 9 = x : 63 \quad x = \frac{5 \times 63}{9} = \frac{315}{9} = 35$$

$$\frac{1}{4} : \frac{7}{8} = 4 : x \quad x = \frac{\frac{7}{8} \times 4}{\frac{1}{4}} = \frac{3\frac{1}{2}}{\frac{1}{4}} = 14$$

If the second and third terms are the same, that number is the *mean proportional* between the other two. Thus, $8 : 4 = 4 : 2$, and 4 is the mean proportional between 8 and 2. The mean proportional between two numbers may be found by multiplying the numbers together and extracting the square root of the product. Thus, the mean proportional between 3 and 12 is found as follows:

$$3 \times 12 = 36 \quad \text{and} \quad \sqrt{36} = 6$$

which is the mean proportional.

Practical Examples Involving Simple Proportion: If it takes 18 days to assemble 4 lathes, how long would it take to assemble 14 lathes?

Let the number of days to be found be x . Then write out the proportion as follows:

$$4 : 18 = 14 : x$$

(lathes : days = lathes : days)

Now find the fourth term by the rule given:

$$x = \frac{18 \times 14}{4} = 63 \text{ days}$$

Thirty-four linear feet of bar stock are required for the blanks for 100 clamping bolts. How many feet of stock would be required for 912 bolts?

Let x = total length of stock required for 912 bolts.

$$34 : 100 = x : 912$$

(feet : bolts = feet : bolts)

Then, the third term $x = (34 \times 912) / 100 = 310$ feet, approximately.

Inverse Proportion: In an inverse proportion, as one of the items involved *increases*, the corresponding item in the proportion *decreases*, or vice versa. For example, a factory employing 270 men completes a given number of typewriters weekly, the number of working hours being 44 per week. How many men would be required for the same production if the working hours were reduced to 40 per week?

The time per week is in an inverse proportion to the number of men employed; the shorter the time, the more men. The inverse proportion is written:

$$270 : x = 40 : 44$$

(men, 44-hour basis: men, 40-hour basis = time, 40-hour basis: time, 44-hour basis)
Thus

$$\frac{270}{x} = \frac{40}{44} \quad \text{and} \quad x = \frac{270 \times 44}{40} = 297 \text{ men}$$

Problems Involving Both Simple and Inverse Proportions: If two groups of data are related both by direct (simple) and inverse proportions among the various quantities, then a simple mathematical relation that may be used in solving problems is as follows:

Product of all directly proportional items in first group
Product of all inversely proportional items in first group

$$= \frac{\text{Product of all directly proportional items in second group}}{\text{Product of all inversely proportional items in second group}}$$

Example: If a man capable of turning 65 studs in a day of 10 hours is paid \$6.50 per hour, how much per hour ought a man be paid who turns 72 studs in a 9-hour day, if compensated in the same proportion?

The first group of data in this problem consists of the number of hours worked by the first man, his hourly wage, and the number of studs which he produces per day; the second group contains similar data for the second man except for his unknown hourly wage, which may be indicated by x .

The labor cost per stud, as may be seen, is directly proportional to the number of hours worked and the hourly wage. These quantities, therefore, are used in the numerators of the fractions in the formula. The labor cost per stud is inversely proportional to the number of studs produced per day. (The greater the number of studs produced in a given time the less the cost per stud.) The numbers of studs per day, therefore, are placed in the denominators of the fractions in the formula. Thus,

$$\frac{10 \times 6.50}{65} = \frac{9 \times x}{72}$$

$$x = \frac{10 \times 6.50 \times 72}{65 \times 9} = \$8.00 \text{ per hour}$$

Percentage.—If out of 100 pieces made, 12 do not pass inspection, it is said that 12 per cent (12 of the hundred) are rejected. If a quantity of steel is bought for \$100 and sold for \$140, the profit is 28.6 per cent of the selling price.

The per cent of gain or loss is found by dividing the amount of gain or loss by the *original* number of which the percentage is wanted, and multiplying the quotient by 100.

Example: Out of a total output of 280 castings a day, 30 castings are, on an average, rejected. What is the percentage of bad castings?

$$\frac{30}{280} \times 100 = 10.7 \text{ per cent}$$

If by a new process 100 pieces can be made in the same time as 60 could formerly be made, what is the gain in output of the new process over the old, expressed in per cent?

Original number, 60; gain $100 - 60 = 40$. Hence,

$$\frac{40}{60} \times 100 = 66.7 \text{ per cent}$$

Care should be taken always to use the original number, or the number of which the percentage is wanted, as the divisor in all percentage calculations. In the example just given, it

is the percentage of gain over the old output 60 that is wanted and not the percentage with relation to the new output too. Mistakes are often made by overlooking this important point.

Fractions

Common Fractions.— Common fractions consist of two basic parts, a denominator, or bottom number, and a numerator, or top number. The denominator shows how many parts the whole unit has been divided into. The numerator indicates the number of parts of the whole that are being considered. A fraction having a value of $\frac{5}{32}$ means the whole unit has been divided into 32 equal parts and 5 of these parts are considered in the value of the fraction.

The following are the basic facts, rules, and definitions concerning common fractions.

A common fraction having the same numerator and denominator is equal to 1. For example, $\frac{2}{2}$, $\frac{4}{4}$, $\frac{8}{8}$, $\frac{16}{16}$, $\frac{32}{32}$, and $\frac{64}{64}$ all equal 1.

Proper Fraction: A proper fraction is a common fraction having a numerator smaller than its denominator, such as $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{7}{64}$.

Improper Fraction: An improper fraction is a common fraction having a numerator larger than its denominator. For example, $\frac{3}{2}$, $\frac{7}{4}$, and $\frac{10}{8}$. To convert a whole number to an improper fraction place the whole number over 1, as in $4 = \frac{4}{1}$ and $3 = \frac{3}{1}$.

Reducible Fraction: A reducible fraction is a common fraction that can be reduced to lower terms. For example, $\frac{2}{4}$ can be reduced to $\frac{1}{2}$, and $\frac{28}{32}$ can be reduced to $\frac{7}{8}$. To reduce a common fraction to lower terms, divide both the numerator and the denominator by the same number. For example, $\frac{28}{32} \div \frac{4}{4} = \frac{7}{8}$ and $\frac{6}{8} \div \frac{2}{2} = \frac{3}{4}$.

Least Common Denominator: A least common denominator is the smallest denominator value that is evenly divisible by the other denominator values in the problem. For example, given the following numbers, $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{8}$, the least common denominator is 8.

Mixed Number: A mixed number is a combination of a whole number and a common fraction, such as $2\frac{1}{2}$, $1\frac{7}{8}$, $3\frac{15}{16}$ and $1\frac{9}{32}$.

To convert mixed numbers to improper fractions, multiply the whole number by the denominator and add the numerator to obtain the new numerator. The denominator remains the same. For example,

$$2\frac{1}{2} = \frac{2 \times 2 + 1}{2} = \frac{5}{2}$$

$$3\frac{7}{16} = \frac{3 \times 16 + 7}{16} = \frac{55}{16}$$

To convert an improper fraction to a mixed number, divide the numerator by the denominator and reduce the remaining fraction to its lowest terms. For example,

$$\frac{17}{8} = 17 \div 8 = 2\frac{1}{8} \text{ and } \frac{26}{16} = 26 \div 16 = 1\frac{10}{16} = 1\frac{5}{8}$$

A fraction may be converted to higher terms by multiplying the numerator and denominator by the same number. For example, $\frac{1}{4}$ in 16ths = $\frac{1}{4} \times \frac{4}{4} = \frac{4}{16}$ and $\frac{3}{8}$ in 32nds = $\frac{3}{8} \times \frac{4}{4} = \frac{12}{32}$.

To change a whole number to a common fraction with a specific denominator value, convert the whole number to a fraction and multiply the numerator and denominator by the desired denominator value.

Example: 4 in 16ths = $\frac{4}{1} \times \frac{16}{16} = \frac{64}{16}$ and 3 in 32nds = $\frac{3}{1} \times \frac{32}{32} = \frac{96}{32}$

Reciprocals.—The reciprocal R of a number N is obtained by dividing 1 by the number; $R = 1/N$. Reciprocals are useful in some calculations because they avoid the use of negative characteristics as in calculations with logarithms and in trigonometry. In trigonometry, the

values *cosecant*, *secant*, and *cotangent* are often used for convenience and are the reciprocals of the *sine*, *cosine*, and *tangent*, respectively (see page 88). The reciprocal of a fraction, for instance $\frac{3}{4}$ is the fraction inverted, since $1 \div \frac{3}{4} = 1 \times \frac{4}{3} = \frac{4}{3}$.

Adding Fractions and Mixed Numbers

To Add Common Fractions: 1) Find and convert to the least common denominator; 2) Add the numerators; 3) Convert the answer to a mixed number, if necessary; and 4) Reduce the fraction to its lowest terms.

To Add Mixed Numbers: 1) Find and convert to the least common denominator; 2) Add the numerators; 3) Add the whole numbers; and 4) Reduce the answer to its lowest terms.

Example, Addition of Common Fractions:

$$\frac{1}{4} + \frac{3}{16} + \frac{7}{8} =$$

$$\frac{1}{4}\left(\frac{4}{4}\right) + \frac{3}{16} + \frac{7}{8}\left(\frac{2}{2}\right) =$$

$$\frac{4}{16} + \frac{3}{16} + \frac{14}{16} = \frac{21}{16}$$

Example, Addition of Mixed Numbers:

$$2\frac{1}{2} + 4\frac{1}{4} + 1\frac{15}{32} =$$

$$2\frac{1}{2}\left(\frac{16}{16}\right) + 4\frac{1}{4}\left(\frac{8}{8}\right) + 1\frac{15}{32} =$$

$$2\frac{16}{32} + 4\frac{8}{32} + 1\frac{15}{32} = 7\frac{39}{32} = 8\frac{7}{32}$$

Subtracting Fractions and Mixed Numbers

To Subtract Common Fractions: 1) Convert to the least common denominator; 2) Subtract the numerators; and 3) Reduce the answer to its lowest terms.

To Subtract Mixed Numbers: 1) Convert to the least common denominator; 2) Subtract the numerators; 3) Subtract the whole numbers; and 4) Reduce the answer to its lowest terms.

Example, Subtraction of Common Fractions:

$$\frac{15}{16} - \frac{7}{32} =$$

$$\frac{15}{16}\left(\frac{2}{2}\right) - \frac{7}{32} =$$

$$\frac{30}{32} - \frac{7}{32} = \frac{23}{32}$$

Example, Subtraction of Mixed Numbers:

$$2\frac{3}{8} - 1\frac{1}{16} =$$

$$2\frac{3}{8}\left(\frac{2}{2}\right) - 1\frac{1}{16} =$$

$$2\frac{6}{16} - 1\frac{1}{16} = 1\frac{5}{16}$$

Multiplying Fractions and Mixed Numbers

To Multiply Common Fractions: 1) Multiply the numerators; 2) Multiply the denominators; and 3) Convert improper fractions to mixed numbers, if necessary.

To Multiply Mixed Numbers: 1) Convert the mixed numbers to improper fractions; 2) Multiply the numerators; 3) Multiply the denominators; and 4) Convert improper fractions to mixed numbers, if necessary.

Example, Multiplication of Common Fractions:

$$\frac{3}{4} \times \frac{7}{16} = \frac{3 \times 7}{4 \times 16} = \frac{21}{64}$$

Example, Multiplication of Mixed Numbers:

$$2\frac{1}{4} \times 3\frac{1}{2} = \frac{9 \times 7}{4 \times 2} = \frac{63}{8} = 7\frac{7}{8}$$

Dividing Fractions and Mixed Numbers

To Divide Common Fractions: 1) Write the fractions to be divided; 2) Invert (switch) the numerator and denominator in the dividing fraction; 3) Multiply the numerators and denominators; and 4) Convert improper fractions to mixed numbers, if necessary.

To Divide Mixed Numbers: 1) Convert the mixed numbers to improper fractions; 2) Write the improper fraction to be divided; 3) Invert (switch) the numerator and denominator in the dividing fraction; 4) Multiplying numerators and denominators; and 5) Convert improper fractions to mixed numbers, if necessary.

Example, Division of Common Fractions:

$$\frac{3}{4} \div \frac{1}{2} = \frac{3 \times 2}{4 \times 1} = \frac{6}{4} = 1\frac{1}{2}$$

Example, Division of Mixed Numbers:

$$2\frac{1}{2} \div 1\frac{7}{8} = \frac{5 \times 8}{2 \times 15} = \frac{40}{30} = 1\frac{1}{3}$$

Decimal Fractions.—Decimal fractions are fractional parts of a whole unit, which have implied denominators that are multiples of 10. A decimal fraction of 0.1 has a value of 1/10th, 0.01 has a value of 1/100th, and 0.001 has a value of 1/1000th. As the number of decimal place values increases, the value of the decimal number changes by a multiple of 10. A single number placed to the right of a decimal point has a value expressed in tenths; two numbers to the right of a decimal point have a value expressed in hundredths; three numbers to the right have a value expressed in thousandths; and four numbers are expressed in ten-thousandths. Since the denominator is implied, the number of decimal places in the numerator indicates the value of the decimal fraction. So a decimal fraction expressed as a 0.125 means the whole unit has been divided into 1000 parts and 125 of these parts are considered in the value of the decimal fraction.

In industry, most decimal fractions are expressed in terms of thousandths rather than tenths or hundredths. So a decimal fraction of 0.2 is expressed as 200 thousandths, not 2 tenths, and a value of 0.75 is expressed as 750 thousandths, rather than 75 hundredths. In the case of four place decimals, the values are expressed in terms of ten-thousandths. So a value of 0.1875 is expressed as 1 thousand 8 hundred and 75 ten-thousandths. When whole numbers and decimal fractions are used together, whole units are shown to the left of a decimal point, while fractional parts of a whole unit are shown to the right.

Example:

10.125	
Whole Units	Fraction Units

Adding Decimal Fractions: 1) Write the problem with all decimal points aligned vertically; 2) Add the numbers as whole number values; and 3) Insert the decimal point in the same vertical column in the answer.

Subtracting Decimal Fractions: 1) Write the problem with all decimal points aligned vertically; 2) Subtract the numbers as whole number values; and 3) Insert the decimal point in the same vertical column in the answer.

Multiplying Decimal Fractions: 1) Write the problem with the decimal points aligned; 2) Multiply the values as whole numbers; 3) Count the number of decimal places in both multiplied values; and 4) Counting from right to left in the answer, insert the decimal point so the number of decimal places in the answer equals the total number of decimal places in the numbers multiplied.

Example, Adding Decimal Fractions:

$$\begin{array}{r} 0.125 \\ 1.0625 \\ 2.50 \\ 0.1875 \\ 3.8750 \\ \hline 7.7500 \end{array}$$

or

$$\begin{array}{r} 1.750 \\ 0.875 \\ 0.125 \\ 2.0005 \\ 4.7505 \end{array}$$

Example, Subtracting Decimal Fractions:

$$\begin{array}{r} 1.750 \\ -0.250 \\ \hline 1.500 \end{array} \quad \text{or} \quad \begin{array}{r} 2.625 \\ -1.125 \\ \hline 1.500 \end{array}$$

Example, Multiplying Decimal Fractions:

0.75		1.625	
<u>0.25</u>		<u>0.033</u>	
375	(four decimal places)	4875	(six decimal places)
<u>150</u>		<u>4875</u>	
0.1875		0.053625	

Continued Fractions.—In dealing with a cumbersome fraction, or one which does not have satisfactory factors, it may be possible to substitute some other, approximately equal, fraction which is simpler or which can be factored satisfactorily. Continued fractions provide a means of computing a series of fractions each of which is a closer approximation to the original fraction than the one preceding it in the series.

A continued fraction is a proper fraction (one whose numerator is smaller than its denominator) expressed in the form shown at the left below; or, it may be convenient to write the left expression as shown at the right below.

$$\frac{N}{D} = \frac{1}{D_1 + \frac{1}{D_2 + \frac{1}{D_3 + \dots}}}$$

$$\frac{N}{D} = \frac{1}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} + \frac{1}{D_4} + \dots$$

The continued fraction is produced from a proper fraction N/D by dividing the numerator N both into itself and into the denominator D . Dividing the numerator into itself gives a result of 1; dividing the numerator into the denominator gives a whole number D_1 plus a remainder fraction R_1 . The process is then repeated on the remainder fraction R_1 to obtain D_2 and R_2 ; then D_3, R_3 , etc., until a remainder of zero results. As an example, using $N/D = 2153/9277$,

$$\frac{2153}{9277} = \frac{2153 \div 2153}{9277 \div 2153} = \frac{1}{4 + \frac{665}{2153}} = \frac{1}{D_1 + R_1}$$

$$R_1 = \frac{665}{2153} = \frac{1}{3 + \frac{158}{665}} = \frac{1}{D_2 + R_2} \text{ etc.}$$

from which it may be seen that $D_1 = 4, R_1 = 665/2153; D_2 = 3, R_2 = 158/665$; and, continuing as was explained previously, it would be found that: $D_3 = 4, R_3 = 33/158; \dots; D_9 = 2, R_9 = 0$. The complete set of continued fraction elements representing $2153/9277$ may then be written as

$$\frac{2153}{9277} = \frac{1}{4 + \frac{1}{3 + \frac{1}{4 + \frac{1}{4 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}}}}}}}$$

$$D_1 \dots \dots \dots D_5 \dots \dots \dots D_9$$

By following a simple procedure, together with a table organized similar to the one below for the fraction $2153/9277$, the denominators D_1, D_2, \dots of the elements of a continued fraction may be used to calculate a series of fractions, each of which is a successively closer approximation, called a *convergent*, to the original fraction N/D .

1) The first row of the table contains column numbers numbered from 1 through 2 plus the number of elements, $2 + 9 = 11$ in this example.

2) The second row contains the denominators of the continued fraction elements in sequence but beginning in column 3 instead of column 1 because columns 1 and 2 must be blank in this procedure.

3) The third row contains the convergents to the original fraction as they are calculated and entered. Note that the fractions $1/0$ and $0/1$ have been inserted into columns 1 and 2. These are two arbitrary convergents, the first equal to infinity, the second to zero, which are used to facilitate the calculations.

4) The convergent in column 3 is now calculated. To find the numerator, multiply the denominator in column 3 by the numerator of the convergent in column 2 and add the numerator of the convergent in column 1. Thus, $4 \times 0 + 1 = 1$.

5) The denominator of the convergent in column 3 is found by multiplying the denominator in column 3 by the denominator of the convergent in column 2 and adding the denominator of the convergent in column 1. Thus, $4 \times 1 + 0 = 4$, and the convergent in column 3 is then $\frac{1}{4}$ as shown in the table.

6) Finding the remaining successive convergents can be reduced to using the simple equation

$$\text{CONVERGENT}_n = \frac{(D_n)(\text{NUM}_{n-1}) + \text{NUM}_{n-2}}{(D_n)(\text{DEN}_{n-1}) + \text{DEN}_{n-2}}$$

in which n = column number in the table; D_n = denominator in column n ; NUM_{n-1} and NUM_{n-2} are numerators and DEN_{n-1} and DEN_{n-2} are denominators of the convergents in the columns indicated by their subscripts; and CONVERGENT_n is the convergent in column n .

Convergents of the Continued Fraction for 2153/9277

Column Number, n	1	2	3	4	5	6	7	8	9	10	11
Denominator, D_n	—	—	4	3	4	4	1	3	1	2	2
Convergent $_n$	$\frac{1}{0}$	$\frac{0}{1}$	$\frac{1}{4}$	$\frac{3}{13}$	$\frac{13}{56}$	$\frac{55}{237}$	$\frac{68}{293}$	$\frac{259}{1116}$	$\frac{327}{1409}$	$\frac{913}{3934}$	$\frac{2153}{9277}$

Notes: The decimal values of the successive convergents in the table are alternately larger and smaller than the value of the original fraction 2153/9277. If the last convergent in the table has the same value as the original fraction 2153/9277, then all of the other calculated convergents are correct.

Conjugate Fractions.—In addition to finding approximate ratios by the use of continued fractions and logarithms of ratios, conjugate fractions may be used for the same purpose, independently, or in combination with the other methods.

Two fractions a/b and c/d are said to be conjugate if $ad - bc = \pm 1$. Examples of such pairs are: $0/1$ and $1/1$; $1/2$ and $1/1$; and $9/10$ and $8/9$. Also, every successive pair of the convergents of a continued fraction are conjugate. Conjugate fractions have certain properties that are useful for solving ratio problems:

1) No fraction between two conjugate fractions a/b and c/d can have a denominator smaller than either b or d .

2) A new fraction, e/f , conjugate to both fractions of a given pair of conjugate fractions, a/b and c/d , and lying between them, may be created by adding respective numerators, $a + c$, and denominators, $b + d$, so that $e/f = (a + c)/(b + d)$.

3) The denominator $f = b + d$ of the new fraction e/f is the smallest of any possible fraction lying between a/b and c/d . Thus, $17/19$ is conjugate to both $8/9$ and $9/10$ and no fraction with denominator smaller than 19 lies between them. This property is important if it is desired to minimize the size of the factors of the ratio to be found.

The following example shows the steps to approximate a ratio for a set of gears to any desired degree of accuracy within the limits established for the allowable size of the factors in the ratio.

Example: Find a set of four change gears, ab/cd , to approximate the ratio 2.105399 accurate to within ± 0.0001 ; no gear is to have more than 120 teeth.

Step 1. Convert the given ratio R to a number r between 0 and 1 by taking its reciprocal: $1/R = 1/2.105399 = 0.4749693 = r$.

Step 2. Select a pair of conjugate fractions a/b and c/d that bracket r . The pair $a/b = 0/1$ and $c/d = 1/1$, for example, will bracket 0.4749693.

Step 3. Add the respective numerators and denominators of the conjugates $0/1$ and $1/1$ to create a new conjugate e/f between 0 and 1: $e/f = (a + c)/(b + d) = (0 + 1)/(1 + 1) = 1/2$.

Step 4. Since 0.4749693 lies between $0/1$ and $1/2$, e/f must also be between $0/1$ and $1/2$: $e/f = (0 + 1)/(1 + 2) = 1/3$.

Step 5. Since 0.4749693 now lies between $1/3$ and $1/2$, e/f must also be between $1/3$ and $1/2$: $e/f = (1 + 1)/(3 + 2) = 2/5$.

Step 6. Continuing as above to obtain successively closer approximations of e/f to 0.4749693, and using a handheld calculator and a scratch pad to facilitate the process, the fractions below, each of which has factors less than 120, were determined:

Fraction	Numerator Factors	Denominator Factors	Error
19/40	19	$2 \times 2 \times 2 \times 5$	+ .000031
28/59	$2 \times 2 \times 7$	59	- .00039
47/99	47	$3 \times 3 \times 11$	- .00022
104/219	$2 \times 2 \times 2 \times 13$	3×73	- .000083
123/259	3×41	7×37	- .000066
142/299	2×71	13×23	- .000053
161/339	7×23	3×113	- .000043
218/459	2×109	$3 \times 3 \times 3 \times 17$	- .000024
256/539	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$	$7 \times 7 \times 11$	- .000016
370/779	$2 \times 5 \times 37$	19×41	- .0000014
759/1598	$3 \times 11 \times 23$	$2 \times 17 \times 47$	- .00000059

Factors for the numerators and denominators of the fractions shown above were found with the aid of the Prime Numbers and Factors tables beginning on page 20. Since in Step 1 the desired ratio of 2.105399 was converted to its reciprocal 0.4749693, all of the above fractions should be inverted. Note also that the last fraction, 759/1598, when inverted to become 1598/759, is in error from the desired value by approximately one-half the amount obtained by trial and error using earlier methods.

Using Continued Fraction Convergents as Conjugates.—Since successive convergents of a continued fraction are also conjugate, they may be used to find a series of additional fractions in between themselves. As an example, the successive convergents 55/237 and 68/293 from the table of convergents for 2153/9277 on page 12 will be used to demonstrate the process for finding the first few in-between ratios.

Desired Fraction $N/D = 2153/9277 = 0.2320793$

a/b	e/f	c/d
(1) $55/237 = .2320675$	^a $123/530 = .2320755$ error = - .0000039	$68/293 = .2320819$
(2) $123/530 = .2320755$	$191/823 = .2320778$ error = - .0000016	$68/293 = .2320819$
(3) $191/823 = .2320778$	^a $259/1116 = .2320789$ error = - .0000005	$68/293 = .2320819$
(4) $259/1116 = .2320789$	$327/1409 = .2320795$ error = + .0000002	$68/293 = .2320819$
(5) $259/1116 = .2320789$	$586/2525 = .2320792$ error = - .0000001	$327/1409 = .2320795$
(6) $586/2525 = .2320792$	$913/3934 = .2320793$ error = - .0000000	$327/1409 = .2320795$

^a Only these ratios had suitable factors below 120.

Step 1. Check the convergents for conjugateness: $55 \times 293 - 237 \times 68 = 16115 - 16116 = -1$ proving the pair to be conjugate.

Step 2. Set up a table as shown above. The leftmost column of line (1) contains the convergent of lowest value, a/b ; the rightmost the higher value, c/d ; and the center column the derived value e/f found by adding the respective numerators and denominators of a/b and c/d . The error or difference between e/f and the desired value N/D , $\text{error} = N/D - e/f$, is also shown.

Step 3. On line (2), the process used on line (1) is repeated with the e/f value from line (1) becoming the new value of a/b while the c/d value remains unchanged. Had the error in e/f been + instead of -, then e/f would have been the new c/d value and a/b would be unchanged.

Step 4. The process is continued until, as seen on line (4), the error changes sign to + from the previous -. When this occurs, the e/f value becomes the c/d value on the next line instead of a/b as previously and the a/b value remains unchanged.

Powers and Roots

The *square* of a number (or quantity) is the product of that number multiplied by itself. Thus, the square of 9 is $9 \times 9 = 81$. The square of a number is indicated by the *exponent* (2), thus: $9^2 = 9 \times 9 = 81$.

The *cube* or *third power* of a number is the product obtained by using that number as a factor three times. Thus, the cube of 4 is $4 \times 4 \times 4 = 64$, and is written 4^3 .

If a number is used as a factor four or five times, respectively, the product is the fourth or fifth power. Thus, $3^4 = 3 \times 3 \times 3 \times 3 = 81$, and $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$. A number can be raised to any power by using it as a factor the required number of times.

The *square root* of a given number is that number which, when multiplied by itself, will give a product equal to the given number. The square root of 16 (written $\sqrt{16}$) equals 4, because $4 \times 4 = 16$.

The *cube root* of a given number is that number which, when used as a factor three times, will give a product equal to the given number. Thus, the cube root of 64 (written $\sqrt[3]{64}$) equals 4, because $4 \times 4 \times 4 = 64$.

The fourth, fifth, etc., roots of a given number are those numbers which when used as factors four, five, etc., times, will give as a product the given number. Thus, $\sqrt[4]{16} = 2$, because $2 \times 2 \times 2 \times 2 = 16$.

In some formulas, there may be such expressions as $(a^2)^3$ and $a^{3/2}$. The first of these, $(a^2)^3$, means that the number a is first to be squared, a^2 , and the result then cubed to give a^6 . Thus, $(a^2)^3$ is equivalent to a^6 which is obtained by *multiplying* the exponents 2 and 3. Similarly, $a^{3/2}$ may be interpreted as the cube of the square root of a , $(\sqrt{a})^3$, or $(a^{1/2})^3$, so that, for example, $16^{3/2} = (\sqrt{16})^3 = 64$.

The multiplications required for raising numbers to powers and the extracting of roots are greatly facilitated by the use of logarithms. Extracting the square root and cube root by the regular arithmetical methods is a slow and cumbersome operation, and any roots can be more rapidly found by using logarithms.

When the power to which a number is to be raised is not an integer, say 1.62, the use of either logarithms or a scientific calculator becomes the only practical means of solution.

Powers of Ten Notation.—Powers of ten notation is used to simplify calculations and ensure accuracy, particularly with respect to the position of decimal points, and also simplifies the expression of numbers which are so large or so small as to be unwieldy. For example, the metric (SI) pressure unit pascal is equivalent to 0.0000986923 atmosphere or 0.0001450377 pound/inch². In powers of ten notation, these figures are 9.86923×10^{-6}

atmosphere and 1.450377×10^{-4} pound/inch². The notation also facilitates adaptation of numbers for electronic data processing and computer readout.

Expressing Numbers in Powers of Ten Notation.—In this system of notation, every number is expressed by two factors, one of which is some integer from 1 to 9 followed by a decimal and the other is some power of 10.

Thus, 10,000 is expressed as 1.0000×10^4 and 10,463 as 1.0463×10^4 . The number 43 is expressed as 4.3×10 and 568 is expressed as 5.68×10^2 .

In the case of decimals, the number 0.0001, which as a fraction is $\frac{1}{10000}$ and is expressed as 1×10^{-4} and 0.0001463 is expressed as 1.463×10^{-4} . The decimal 0.498 is expressed as 4.98×10^{-1} and 0.03146 is expressed as 3.146×10^{-2} .

Rules for Converting Any Number to Powers of Ten Notation.—Any number can be converted to the powers of ten notation by means of one of two rules.

Rule 1: If the number is a whole number or a whole number and a decimal so that it has digits to the left of the decimal point, the decimal point is moved a sufficient number of places to the *left* to bring it to the immediate right of the first digit. With the decimal point shifted to this position, the number so written comprises the *first* factor when written in powers of ten notation.

The number of places that the decimal point is moved to the left to bring it immediately to the right of the first digit is the *positive* index or power of 10 that comprises the *second* factor when written in powers of ten notation.

Thus, to write 4639 in this notation, the decimal point is moved three places to the left giving the two factors: 4.639×10^3 . Similarly,

$$431.412 = 4.31412 \times 10^2 \quad 986388 = 9.86388 \times 10^5$$

Rule 2: If the number is a decimal, i.e., it has digits entirely to the right of the decimal point, then the decimal point is moved a sufficient number of places to the *right* to bring it immediately to the right of the first digit. With the decimal point shifted to this position, the number so written comprises the *first* factor when written in powers of ten notation.

The number of places that the decimal point is moved to the *right* to bring it immediately to the right of the first digit is the *negative* index or power of 10 that follows the number when written in powers of ten notation.

Thus, to bring the decimal point in 0.005721 to the immediate right of the first digit, which is 5, it must be moved *three* places to the right, giving the two factors: 5.721×10^{-3} . Similarly,

$$0.469 = 4.69 \times 10^{-1} \quad 0.0000516 = 5.16 \times 10^{-5}$$

Multiplying Numbers Written in Powers of Ten Notation.—When multiplying two numbers written in the powers of ten notation together, the procedure is as follows:

1) Multiply the first factor of one number by the first factor of the other to obtain the first factor of the product.

2) Add the index of the second factor (which is some power of 10) of one number to the index of the second factor of the other number to obtain the index of the second factor (which is some power of 10) in the product. Thus:

$$(4.31 \times 10^{-2}) \times (9.0125 \times 10) = (4.31 \times 9.0125) \times 10^{-2+1} = 38.844 \times 10^{-1}$$

$$(5.986 \times 10^4) \times (4.375 \times 10^3) = (5.986 \times 4.375) \times 10^{4+3} = 26.189 \times 10^7$$

In the preceding calculations, neither of the results shown are in the conventional powers of ten form since the first factor in each has two digits. In the conventional powers of ten notation, the results would be

$38.844 \times 10^{-1} = 3.884 \times 10^0 = 3.884$, since $10^0 = 1$, and $26.189 \times 10^7 = 2.619 \times 10^8$ in each case rounding off the first factor to three decimal places.

When multiplying several numbers written in this notation together, the procedure is the same. All of the first factors are multiplied together to get the first factor of the product and all of the indices of the respective powers of ten are added together, taking into account their respective signs, to get the index of the second factor of the product. Thus, $(4.02 \times 10^{-3}) \times (3.987 \times 10) \times (4.863 \times 10^5) = (4.02 \times 3.987 \times 4.863) \times 10^{(-3+1+5)} = 77.94 \times 10^3 = 7.79 \times 10^4$ rounding off the first factor to two decimal places.

Dividing Numbers Written in Powers of Ten Notation.—When dividing one number by another when both are written in this notation, the procedure is as follows:

1) Divide the first factor of the dividend by the first factor of the divisor to get the first factor of the quotient.

2) Subtract the index of the second factor of the divisor from the index of the second factor of the dividend, taking into account their respective signs, to get the index of the second factor of the quotient. Thus:

$$(4.31 \times 10^{-2}) \div (9.0125 \times 10) =$$

$$(4.31 \div 9.0125) \times (10^{-2-1}) = 0.4782 \times 10^{-3} = 4.782 \times 10^{-4}$$

It can be seen that this system of notation is helpful where several numbers of different magnitudes are to be multiplied and divided.

Example: Find the quotient of $\frac{250 \times 4698 \times 0.00039}{43678 \times 0.002 \times 0.0147}$

Solution: Changing all these numbers to powers of ten notation and performing the operations indicated:

$$\begin{aligned} \frac{(2.5 \times 10^2) \times (4.698 \times 10^3) \times (3.9 \times 10^{-4})}{(4.3678 \times 10^4) \times (2 \times 10^{-3}) \times (1.47 \times 10^{-2})} &= \\ &= \frac{(2.5 \times 4.698 \times 3.9)(10^{2+3-4})}{(4.3678 \times 2 \times 1.47)(10^{4-3-2})} = \frac{45.8055 \times 10}{12.8413 \times 10^{-1}} \\ &= 3.5670 \times 10^{1-(-1)} = 3.5670 \times 10^2 = 356.70 \end{aligned}$$

Constants Frequently Used in Mathematical Expressions

$0.00872665 = \frac{\pi}{360}$	$0.8660254 = \frac{\sqrt{3}}{2}$	$2.0943951 = \frac{2\pi}{3}$	$4.712389 = \frac{3\pi}{2}$
$0.01745329 = \frac{\pi}{180}$	$1.0471975 = \frac{\pi}{3}$	$2.3561945 = \frac{3\pi}{4}$	$5.2359878 = \frac{5\pi}{3}$
$0.26179939 = \frac{\pi}{12}$	$1.1547005 = \frac{2\sqrt{3}}{3}$	$2.5980762 = \frac{3\sqrt{3}}{2}$	$5.4977871 = \frac{7\pi}{4}$
$0.39269908 = \frac{\pi}{8}$	$1.2247449 = \frac{\sqrt{3}}{\sqrt{2}}$	$2.6179939 = \frac{5\pi}{6}$	$5.7595865 = \frac{11\pi}{6}$
$0.52359878 = \frac{\pi}{6}$	$1.4142136 = \sqrt{2}$	$3.1415927 = \pi$	$6.2831853 = 2\pi$
$0.57735027 = \frac{\sqrt{3}}{3}$	$1.5707963 = \frac{\pi}{2}$	$3.6651914 = \frac{7\pi}{6}$	$9.8696044 = \pi^2$
$0.62035049 = \sqrt[3]{\frac{3}{4\pi}}$	$1.7320508 = \sqrt{3}$	$3.9269908 = \frac{5\pi}{4}$	$12.566371 = 4\pi$
$0.78539816 = \frac{\pi}{4}$	$2.4674011 = \frac{\pi^2}{4}$	$4.1887902 = \frac{4\pi}{3}$	$57.29578 = \frac{180}{\pi}$
			$114.59156 = \frac{360}{\pi}$

Imaginary and Complex Numbers

Complex or Imaginary Numbers.—Complex or imaginary numbers represent a class of mathematical objects that are used to simplify certain problems, such as the solution of polynomial equations. The basis of the complex number system is the unit imaginary number i that satisfies the following relations:

$$i^2 = (-i)^2 = -1 \quad i = \sqrt{-1} \quad -i = -\sqrt{-1}$$

In electrical engineering and other fields, the unit imaginary number is often represented by j rather than i . However, the meaning of the two terms is identical.

Rectangular or Trigonometric Form: Every complex number, Z , can be written as the sum of a real number and an imaginary number. When expressed as a sum, $Z = a + bi$, the complex number is said to be in rectangular or trigonometric form. The real part of the number is a , and the imaginary portion is bi because it has the imaginary unit assigned to it.

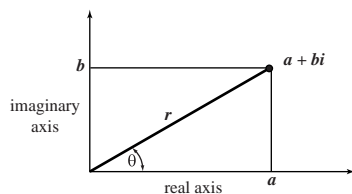
Polar Form: A complex number $Z = a + bi$ can also be expressed in polar form, also known as phasor form. In polar form, the complex number Z is represented by a magnitude r and an angle θ as follows:

$$Z = r \angle \theta$$

$\angle \theta$ = a direction, the angle whose tangent is $b \div a$, thus $\theta = \text{atan} \frac{b}{a}$ and

$$r = \sqrt{a^2 + b^2} \text{ is the magnitude}$$

A complex number can be plotted on a real-imaginary coordinate system known as the complex plane. The figure below illustrates the relationship between the rectangular coordinates a and b , and the polar coordinates r and θ .



Complex Number in the Complex Plane

The rectangular form can be determined from r and θ as follows:

$$a = r \cos \theta \quad b = r \sin \theta \quad a + bi = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta)$$

The rectangular form can also be written using Euler's Formula:

$$e^{\pm i\theta} = \cos \theta \pm i \sin \theta \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Complex Conjugate: Complex numbers commonly arise in finding the solution of polynomials. A polynomial of n^{th} degree has n solutions, an even number of which are complex and the rest are real. The complex solutions always appear as complex conjugate pairs in the form $a + bi$ and $a - bi$. The product of these two conjugates, $(a + bi) \times (a - bi) = a^2 + b^2$, is the square of the magnitude r illustrated in the previous figure.

Operations on Complex Numbers

Example 1, Addition: When adding two complex numbers, the real parts and imaginary parts are added separately, the real parts added to real parts and the imaginary to imaginary parts. Thus,

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