# Introduction to Real Analysis

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TO BEVERLY

# Contents

# Preface

This is a text for a two-term course in introductory real analysis for junior or senior mathematics majors and science students with a serious interest in mathematics. Prospective educators or mathematically gifted high school students can also benefit from the mathematical maturity that can be gained from an introductory real analysis course.

The book is designed to fill the gaps left in the development of calculus as it is usually presented in an elementary course, and to provide the background required for insight into more advanced courses in pure and applied mathematics. The standard elementary calculus sequence is the only specific prerequisite for Chapters 1–5, which deal with real-valued functions. (However, other analysis oriented courses, such as elementary differential equation, also provide useful preparatory experience.) Chapters 6 and 7 require a working knowledge of determinants, matrices and linear trnsformations, typically available from a first course in linear algebra. Chapter 8 is accessible after completion of Chapters 1–5.

Without taking a position for or against the current reforms in mathematics teaching, I think it is fair to say that the transition from elementary courses such as calculus, linear algebra, and differential equations to a rigorous real analysis course is a bigger step today than it was just a few year ago. To make this step today's students need more help than their predecessors did, and must be coached and encouraged more. Therefore, while striving throughout to maintain a high level of rigor, I have tried to write as clearly and informally as possible. In this connection I find it useful to address the student in the second person. I have included 295 completely worked out examples to illustrate and clarify all major theorems and definitions.

I have emphasized careful statements of definitions and theorems and have tried to be complete and detailed in proofs, except for omissions left to exercises. I give a thorough treatment of real-valued functions before considering vector-valued functions. In making the transition from one to several variables and from real-valued to vector-valued functions, I have left to the student some proofs that are essentially repetitions of earlier theorems. I believe that working through the details of straightforward generalizations of more elementary results is good practice for the student.

Great care has gone into the preparation of the 760 numbered exercises, many with multiple parts. They range from routine to very difficult. Hints are provided for the more difficult parts of the exercises.

## Organization

Chapter 1 is concerned with the real number system. Section 1.1 begins with a brief discussion of the axioms for a complete ordered field, but no attempt is made to develop the reals from them; rather, it is assumed that the student is familiar with the consequences of these axioms, except for one: completeness. Since the difference between a rigorous and nonrigorous treatment of calculus can be described largely in terms of the attitude taken toward completeness, I have devoted considerable effort to developing its consequences. Section 1.2 is about induction. Although this may seem out of place in a real analysis course, I have found that the typical beginning real analysis student simply cannot do an induction proof without reviewing the method. Section 1.3 is devoted to elementary set theory and the topology of the real line, ending with the Heine-Borel and Bolzano-Weierstrass theorems.

Chapter 2 covers the differential calculus of functions of one variable: limits, continuity, differentiablility, L'Hospital's rule, and Taylor's theorem. The emphasis is on rigorous presentation of principles; no attempt is made to develop the properties of specific elementary functions. Even though this may not be done rigorously in most contemporary calculus courses, I believe that the student's time is better spent on principles rather than on reestablishing familiar formulas and relationships.

Chapter 3 is to devoted to the Riemann integral of functions of one variable. In Section 3.1 the integral is defined in the standard way in terms of Riemann sums. Upper and lower integrals are also defined there and used in Section 3.2 to study the existence of the integral. Section 3.3 is devoted to properties of the integral. Improper integrals are studied in Section 3.4. I believe that my treatment of improper integrals is more detailed than in most comparable textbooks. A more advanced look at the existence of the proper Riemann integral is given in Section 3.5, which concludes with Lebesgue's existence criterion. This section can be omitted without compromising the student's preparedness for subsequent sections.

Chapter 4 treats sequences and series. Sequences of constant are discussed in Section 4.1. I have chosen to make the concepts of limit inferior and limit superior parts of this development, mainly because this permits greater flexibility and generality, with little extra effort, in the study of infinite series. Section 4.2 provides a brief introduction to the way in which continuity and differentiability can be studied by means of sequences. Sections 4.3–4.5 treat infinite series of constant, sequences and infinite series of functions, and power series, again in greater detail than in most comparable textbooks. The instructor who chooses not to cover these sections completely can omit the less standard topics without loss in subsequent sections.

Chapter 5 is devoted to real-valued functions of several variables. It begins with a discussion of the toplogy of $\mathbb{R}^n$ in Section 5.1. Continuity and differentiability are discussed in Sections 5.2 and 5.3. The chain rule and Taylor's theorem are discussed in Section 5.4.

Chapter 6 covers the differential calculus of vector-valued functions of several variables. Section 6.1 reviews matrices, determinants, and linear transformations, which are integral parts of the differential calculus as presented here. In Section 6.2 the differential of a vector-valued function is defined as a linear transformation, and the chain rule is discussed in terms of composition of such functions. The inverse function theorem is the subject of Section 6.3, where the notion of branches of an inverse is introduced. In Section 6.4 the implicit function theorem is motivated by first considering linear transformations and then stated and proved in general.

Chapter 7 covers the integral calculus of real-valued functions of several variables. Multiple integrals are defined in Section 7.1, first over rectangular parallelepipeds and then over more general sets. The discussion deals with the multiple integral of a function whose discontinuities form a set of Jordan content zero. Section 7.2 deals with the evaluation by iterated integrals. Section 7.3 begins with the definition of Jordan measurability, followed by a derivation of the rule for change of content under a linear transformation, an intuitive formulation of the rule for change of variables in multiple integrals, and finally a careful statement and proof of the rule. The proof is complicated, but this is unavoidable.

Chapter 8 deals with metric spaces. The concept and properties of a metric space are introduced in Section 8.1. Section 8.2 discusses compactness in a metric space, and Section 8.3 discusses continuous functions on metric spaces.

Although this book has been published previously in hard copy, this electronic edition should be regarded as a first edition, since producing it involved the nontrivial task of combining LaTeX files that were originally submitted to the publisher separately, and introducing new fonts. Hence, there are undoubtedly errors–mathematical and typographical–in this edition. Corrections are welcome and will be incorporated when received.

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# CHAPTER 1

## The Real Numbers

IN THIS CHAPTER we begin the study of the real number system. The concepts discussed here will be used throughout the book.

SECTION 1.1 deals with the axioms that define the real numbers, definitions based on them, and some basic properties that follow from them.

SECTION 1.2 emphasizes the principle of mathematical induction.

SECTION 1.3 introduces basic ideas of set theory in the context of sets of real numbers. In this section we prove two fundamental theorems: the Heine–Borel and Bolzano–Weierstrass theorems.

## 1.1 THE REAL NUMBER SYSTEM

Having taken calculus, you know a lot about the real number system; however, you probably do not know that all its properties follow from a few basic ones. Although we will not carry out the development of the real number system from these basic properties, it is useful to state them as a starting point for the study of real analysis and also to focus on one property, completeness, that is probably new to you.

### Field Properties

The real number system (which we will often call simply the *reals*) is first of all a set $\{a, b, c, \dots\}$ on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

**(A)** $a + b = b + a$ and $ab = ba$ (commutative laws).

**(B)** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$ (associative laws).

**(C)** $a(b + c) = ab + ac$ (distributive law).

**(D)** There are distinct real numbers 0 and 1 such that $a + 0 = a$ and $a1 = a$ for all $a$.

**(E)** For each $a$ there is a real number $-a$ such that $a + (-a) = 0$, and if $a \neq 0$, there is a real number $1/a$ such that $a(1/a) = 1$.

The manipulative properties of the real numbers, such as the relations

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(3a+2b)(4c+2d) = 12ac + 6ad + 8bc + 4bd,$$
$$(-a) = (-1)a, \quad a(-b) = (-a)b = -ab,$$

and

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad (b, d \neq 0),$$

all follow from **(A)**–**(E)**. We assume that you are familiar with these properties.

A set on which two operations are defined so as to have properties **(A)**–**(E)** is called a *field*. The real number system is by no means the only field. The *rational numbers* (which are the real numbers that can be written as $r = p/q$, where $p$ and $q$ are integers and $q \neq 0$) also form a field under addition and multiplication. The simplest possible field consists of two elements, which we denote by 0 and 1, with addition defined by

$$0 + 0 = 1 + 1 = 0, \quad 1 + 0 = 0 + 1 = 1, \tag{1}$$

and multiplication defined by

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1 \tag{2}$$

(Exercise 2).

## The Order Relation

The real number system is ordered by the relation $<$, which has the following properties.

**(F)** For each pair of real numbers $a$ and $b$, exactly one of the following is true:

$$a = b, \quad a < b, \quad \text{or} \quad b < a.$$

**(G)** If $a < b$ and $b < c$, then $a < c$. (The relation $<$ is *transitive*.)

**(H)** If $a < b$, then $a + c < b + c$ for any $c$, and if $0 < c$, then $ac < bc$.

A field with an order relation satisfying **(F)**–**(H)** is an *ordered* field. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field, but it is impossible to define an order on the field with two elements defined by (1) and (2) so as to make it into an ordered field (Exercise 2).

We assume that you are familiar with other standard notation connected with the order relation: thus, $a > b$ means that $b < a$; $a \geq b$ means that either $a = b$ or $a > b$; $a \leq b$ means that either $a = b$ or $a < b$; the *absolute value of* $a$, denoted by $|a|$, equals $a$ if $a \geq 0$ or $-a$ if $a \leq 0$. (Sometimes we call $|a|$ the *magnitude* of $a$.)

You probably know the following theorem from calculus, but we include the proof for your convenience.

**Theorem 1.1.1 (The Triangle Inequality)** *If $a$ and $b$ are any two real numbers, then*

$$|a+b| \le |a| + |b|. \tag{3}$$

**Proof** There are four possibilities:

**(a)** If $a \ge 0$ and $b \ge 0$, then $a + b \ge 0$, so $|a+b| = a + b = |a| + |b|$.

**(b)** If $a \le 0$ and $b \le 0$, then $a + b \le 0$, so $|a+b| = -a + (-b) = |a| + |b|$.

**(c)** If $a \le 0$ and $b \ge 0$, then $a + b = -|a| + |b|$, so $|a+b| = \big|-|a| + |b|\big| \le |a| + |b|$.

**(d)** If $a \ge 0$ and $b \le 0$, then $a + b = |a| - |b|$, so $|a+b| = \big||a| - |b|\big| \le |a| + |b|$. ☐

The triangle inequality appears in various forms in many contexts. It is the most important inequality in mathematics. We will use it often.

**Corollary 1.1.2** *If $a$ and $b$ are any two real numbers, then*

$$|a-b| \ge \big||a| - |b|\big| \tag{4}$$

*and*

$$|a+b| \ge \big||a| - |b|\big|. \tag{5}$$

**Proof** Replacing $a$ by $a - b$ in (3) yields

$$|a| \le |a-b| + |b|,$$

so

$$|a-b| \ge |a| - |b|. \tag{6}$$

Interchanging $a$ and $b$ here yield

$$|b-a| \ge |b| - |a|,$$

which is equivalent to

$$|a-b| \ge |b| - |a|, \tag{7}$$

since $|b-a| = |a-b|$. Since

$$\big||a| - |b|\big| = \begin{cases} |a| - |b| & \text{if} \quad |a| > |b|, \\ |b| - |a| & \text{if} \quad |b| > |a|, \end{cases}$$

(6) and (7) imply (4). Replacing $b$ by $-b$ in (4) yields (5), since $|-b| = |b|$. ☐

## Supremum of a Set

A set $S$ of real numbers is *bounded above* if there is a real number $b$ such that $x \le b$ whenever $x \in S$. In this case, $b$ is an *upper bound* of $S$. If $b$ is an upper bound of $S$, then so is any larger number, because of property **(G)**. If $\beta$ is an upper bound of $S$, but no number less than $\beta$ is, then $\beta$ is a *supremum* of $S$, and we write

$$\beta = \sup S.$$

With the real numbers associated in the usual way with the points on a line, these definitions can be interpreted geometrically as follows: $b$ is an upper bound of $S$ if no point of $S$ is to the right of $b$; $\beta = \sup S$ if no point of $S$ is to the right of $\beta$, but there is at least one point of $S$ to the right of any number less than $\beta$ (Figure 1.1.1).

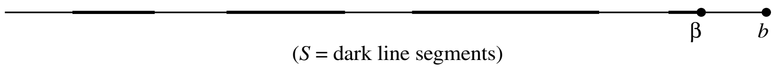


**Figure 1.1.1**

**Example 1.1.1** If $S$ is the set of negative numbers, then any nonnegative number is an upper bound of $S$, and $\sup S = 0$. If $S_1$ is the set of negative integers, then any number $a$ such that $a \geq -1$ is an upper bound of $S_1$, and $\sup S_1 = -1$. ■

This example shows that a supremum of a set may or may not be in the set, since $S_1$ contains its supremum, but $S$ does not.

A *nonempty* set is a set that has at least one member. The *empty set*, denoted by $\emptyset$, is the set that has no members. Although it may seem foolish to speak of such a set, we will see that it is a useful idea.

## The Completeness Axiom

It is one thing to define an object and another to show that there really is an object that satisfies the definition. (For example, does it make sense to define the smallest positive real number?) This observation is particularly appropriate in connection with the definition of the supremum of a set. For example, the empty set is bounded above by every real number, so it has no supremum. (Think about this.) More importantly, we will see in Example 1.1.2 that properties **(A)**–**(H)** do not guarantee that every nonempty set that is bounded above has a supremum. Since this property is indispensable to the rigorous development of calculus, we take it as an axiom for the real numbers.

**(I)** If a nonempty set of real numbers is bounded above, then it has a supremum.

Property **(I)** is called *completeness*, and we say that the real number system is a *complete ordered field.* It can be shown that the real number system is essentially the only complete ordered field; that is, if an alien from another planet were to construct a mathematical system with properties **(A)**–**(I)**, the alien's system would differ from the real number system only in that the alien might use different symbols for the real numbers and $+$, $\cdot$, and $<$.

**Theorem 1.1.3** *If a nonempty set $S$ of real numbers is bounded above, then* $\sup S$ *is the unique real number $\beta$ such that*

**(a)** *$x \leq \beta$ for all $x$ in $S$;*

**(b)** *if $\epsilon > 0$ (no matter how small), there is an $x_0$ in $S$ such that $x_0 > \beta - \epsilon$.*

**Proof** We first show that $\beta = \sup S$ has properties **(a)** and **(b)**. Since $\beta$ is an upper bound of $S$, it must satisfy **(a)**. Since any real number $a$ less than $\beta$ can be written as $\beta - \epsilon$ with $\epsilon = \beta - a > 0$, **(b)** is just another way of saying that no number less than $\beta$ is an upper bound of $S$. Hence, $\beta = \sup S$ satisfies **(a)** and **(b)**.

Now we show that there cannot be more than one real number with properties **(a)** and **(b)**. Suppose that $\beta_1 < \beta_2$ and $\beta_2$ has property **(b)**; thus, if $\epsilon > 0$, there is an $x_0$ in $S$ such that $x_0 > \beta_2 - \epsilon$. Then, by taking $\epsilon = \beta_2 - \beta_1$, we see that there is an $x_0$ in $S$ such that

$$x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1,$$

so $\beta_1$ cannot have property **(a)**. Therefore, there cannot be more than one real number that satisfies both **(a)** and **(b)**. □

## Some Notation

We will often define a set $S$ by writing $S = \{x \mid \cdots\}$, which means that $S$ consists of all $x$ that satisfy the conditions to the right of the vertical bar; thus, in Example 1.1.1,

$$S = \{x \mid x < 0\} \tag{8}$$

and

$$S_1 = \{x \mid x \text{ is a negative integer}\}.$$

We will sometimes abbreviate "$x$ is a member of $S$" by $x \in S$, and "$x$ is not a member of $S$" by $x \notin S$. For example, if $S$ is defined by (8), then

$$-1 \in S \quad \text{but} \quad 0 \notin S.$$

A *nonempty* set is a set that has at least one member. The *empty set* , denoted by $\emptyset$, is the set that has no members. Although it may seem foolish to speak of such a set, we will see that it is a useful concept.

## The Archimedean Property

The property of the real numbers described in the next theorem is called the *Archimedean property*. Intuitively, it states that it is possible to exceed any positive number, no matter how large, by adding an arbitrary positive number, no matter how small, to itself sufficiently many times.

**Theorem 1.1.4 (The Archimedean Property)** *If $\rho$ and $\epsilon$ are positive, then $n\epsilon > \rho$ for some integer $n$.*

**Proof** The proof is by contradiction. If the statement is false, $\rho$ is an upper bound of the set

$$S = \{x \mid x = n\epsilon, n \text{ is an integer}\}.$$

Therefore, $S$ has a supremum $\beta$, by property **(I)**. Therefore,

$$n\epsilon \le \beta \quad \text{for all integers } n. \tag{9}$$

Since $n+1$ is an integer whenever $n$ is, (9) implies that

$$(n+1)\epsilon \le \beta$$

and therefore

$$n\epsilon \le \beta - \epsilon$$

for all integers $n$. Hence, $\beta - \epsilon$ is an upper bound of $S$. Since $\beta - \epsilon < \beta$, this contradicts the definition of $\beta$. $\square$

## Density of the Rationals and Irrationals

**Definition 1.1.5** A set $D$ is *dense in the reals* if every open interval $(a, b)$ contains a member of $D$. $\blacksquare$

**Theorem 1.1.6** *The rational numbers are dense in the reals; that is, if $a$ and $b$ are real numbers with $a < b$, there is a rational number $p/q$ such that $a < p/q < b$.*

**Proof** From Theorem 1.1.4 with $\rho = 1$ and $\epsilon = b - a$, there is a positive integer $q$ such that $q(b-a) > 1$. There is also an integer $j$ such that $j > qa$. This is obvious if $a \le 0$, and it follows from Theorem 1.1.4 with $\epsilon = 1$ and $\rho = qa$ if $a > 0$. Let $p$ be the smallest integer such that $p > qa$. Then $p - 1 \le qa$, so

$$qa < p \le qa + 1.$$

Since $1 < q(b-a)$, this implies that

$$qa < p < qa + q(b-a) = qb,$$

so $qa < p < qb$. Therefore, $a < p/q < b$. $\square$

**Example 1.1.2** The rational number system is not complete; that is, a set of rational numbers may be bounded above (by rationals), but not have a rational upper bound less than any other rational upper bound. To see this, let

$$S = \left\{ r \;\middle|\; r \text{ is rational and } r^2 < 2 \right\}.$$

If $r \in S$, then $r < \sqrt{2}$. Theorem 1.1.6 implies that if $\epsilon > 0$ there is a rational number $r_0$ such that $\sqrt{2} - \epsilon < r_0 < \sqrt{2}$, so Theorem 1.1.3 implies that $\sqrt{2} = \sup S$. However, $\sqrt{2}$ is *irrational*; that is, it cannot be written as the ratio of integers (Exercise 3). Therefore, if $r_1$ is any rational upper bound of $S$, then $\sqrt{2} < r_1$. By Theorem 1.1.6, there is a rational number $r_2$ such that $\sqrt{2} < r_2 < r_1$. Since $r_2$ is also a rational upper bound of $S$, this shows that $S$ has no rational supremum. $\blacksquare$

Since the rational numbers have properties **(A)**–**(H)**, but not **(I)**, this example shows that **(I)** does not follow from **(A)**–**(H)**.

**Theorem 1.1.7** *The set of irrational numbers is dense in the reals; that is, if $a$ and $b$ are real numbers with $a < b$, there is an irrational number $t$ such that $a < t < b$.*

**Proof** From Theorem 1.1.6, there are rational numbers $r_1$ and $r_2$ such that

$$a < r_1 < r_2 < b. \tag{10}$$

Let

$$t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1).$$

Then $t$ is irrational (why?) and $r_1 < t < r_2$, so $a < t < b$, from (10). ☐

### Infimum of a Set

A set $S$ of real numbers is *bounded below* if there is a real number $a$ such that $x \geq a$ whenever $x \in S$. In this case, $a$ is a *lower bound* of $S$. If $a$ is a lower bound of $S$, so is any smaller number, because of property **(G)**. If $\alpha$ is a lower bound of $S$, but no number greater than $\alpha$ is, then $\alpha$ is an *infimum* of $S$, and we write

$$\alpha = \inf S.$$

Geometrically, this means that there are no points of $S$ to the left of $\alpha$, but there is at least one point of $S$ to the left of any number greater than $\alpha$.

**Theorem 1.1.8** *If a nonempty set $S$ of real numbers is bounded below, then* $\inf S$ *is the unique real number $\alpha$ such that*

**(a)** $x \geq \alpha$ *for all $x$ in $S$;*

**(b)** *if* $\epsilon > 0$ (*no matter how small*), *there is an $x_0$ in $S$ such that $x_0 < \alpha + \epsilon$.*

**Proof** (Exercise 6)

A set $S$ is *bounded* if there are numbers $a$ and $b$ such that $a \leq x \leq b$ for all $x$ in $S$. A bounded nonempty set has a unique supremum and a unique infimum, and

$$\inf S \leq \sup S \tag{11}$$

(Exercise 7). ☐

### The Extended Real Number System

A nonempty set $S$ of real numbers is *unbounded above* if it has no upper bound, or *unbounded below* if it has no lower bound. It is convenient to adjoin to the real number system two fictitious points, $+\infty$ (which we usually write more simply as $\infty$) and $-\infty$, and to define the order relationships between them and any real number $x$ by

$$-\infty < x < \infty. \tag{12}$$

We call $\infty$ and $-\infty$ *points at infinity*. If $S$ is a nonempty set of reals, we write

$$\sup S = \infty \tag{13}$$

to indicate that $S$ is unbounded above, and

$$\inf S = -\infty \tag{14}$$

to indicate that $S$ is unbounded below.

**Example 1.1.3** If

$$S = \{x \mid x < 2\},$$

then $\sup S = 2$ and $\inf S = -\infty$. If

$$S = \{x \mid x \geq -2\},$$

then $\sup S = \infty$ and $\inf S = -2$. If $S$ is the set of all integers, then $\sup S = \infty$ and $\inf S = -\infty$. ■

The real number system with $\infty$ and $-\infty$ adjoined is called the *extended real number system*, or simply the *extended reals*. A member of the extended reals differing from $-\infty$ and $\infty$ is *finite*; that is, an ordinary real number is finite. However, the word "finite" in "finite real number" is redundant and used only for emphasis, since we would never refer to $\infty$ or $-\infty$ as real numbers.

The arithmetic relationships among $\infty$, $-\infty$, and the real numbers are defined as follows.

**(a)** If $a$ is any real number, then

$$\begin{aligned} a + \infty &= \quad \infty + a = \quad \infty, \\ a - \infty &= -\infty + a = -\infty, \\ \frac{a}{\infty} &= \frac{a}{-\infty} = 0. \end{aligned}$$

**(b)** If $a > 0$, then

$$\begin{aligned} a\,\infty \quad &= \quad \infty\, a \quad = \quad \infty, \\ a\,(-\infty) &= (-\infty)\, a = -\infty. \end{aligned}$$

**(c)** If $a < 0$, then

$$\begin{aligned} a\,\infty \quad &= \quad \infty\, a \quad = -\infty, \\ a\,(-\infty) &= (-\infty)\, a = \quad \infty. \end{aligned}$$

We also define

$$\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$$

and

$$-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty.$$

Finally, we define

$$|\infty| = |-\infty| = \infty.$$

The introduction of $\infty$ and $-\infty$, along with the arithmetic and order relationships defined above, leads to simplifications in the statements of theorems. For example, the inequality (11), first stated only for bounded sets, holds for any nonempty set $S$ if it is interpreted properly in accordance with (12) and the definitions of (13) and (14). Exercises 10**(b)** and 11**(b)** illustrate the convenience afforded by some of the arithmetic relationships with extended reals, and other examples will illustrate this further in subsequent sections.

It is not useful to define $\infty - \infty$, $0 \cdot \infty$, $\infty/\infty$, and $0/0$. They are called *indeterminate forms*, and left undefined. You probably studied indeterminate forms in calculus; we will look at them more carefully in Section 2.4.

## 1.1 Exercises

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**1.** Write the following expressions in equivalent forms not involving absolute values.

**(a)** $a + b + |a - b|$ **(b)** $a + b - |a - b|$

**(c)** $a + b + 2c + |a - b| + \big|a + b - 2c + |a - b|\big|$

**(d)** $a + b + 2c - |a - b| - \big|a + b - 2c - |a - b|\big|$

**2.** Verify that the set consisting of two members, 0 and 1, with operations defined by Eqns. (1) and (2), is a field. Then show that it is impossible to define an order $<$ on this field that has properties **(F)**, **(G)**, and **(H)**.

**3.** Show that $\sqrt{2}$ is irrational. HINT: *Show that if* $\sqrt{2} = m/n$, *where* $m$ *and* $n$ *are integers*, *then both* $m$ *and* $n$ *must be even*. *Obtain a contradiction from this*.

**4.** Show that $\sqrt{p}$ is irrational if $p$ is prime.

**5.** Find the supremum and infimum of each $S$. State whether they are in $S$.

**(a)** $S = \left\{x \,\middle|\, x = -(1/n) + [1 + (-1)^n]\, n^2, n \geq 1\right\}$
**(b)** $S = \left\{x \,\middle|\, x^2 < 9\right\}$
**(c)** $S = \left\{x \,\middle|\, x^2 \leq 7\right\}$
**(d)** $S = \left\{x \,\middle|\, |2x + 1| < 5\right\}$
**(e)** $S = \left\{x \,\middle|\, (x^2 + 1)^{-1} > \frac{1}{2}\right\}$
**(f)** $S = \left\{x \,\middle|\, x = \text{rational and } x^2 \leq 7\right\}$

**6.** Prove Theorem 1.1.8. HINT: *The set* $T = \left\{x \,\middle|\, -x \in S\right\}$ *is bounded above if* $S$ *is bounded below*. *Apply property* **(I)** *and Theorem* 1.1.3 *to* $T$.

**7.** **(a)** Show that

$$\inf S \leq \sup S \tag{A}$$

for any nonempty set $S$ of real numbers, and give necessary and sufficient conditions for equality.

**(b)** Show that if $S$ is unbounded then (A) holds if it is interpreted according to Eqn. (12) and the definitions of Eqns. (13) and (14).

**8.** Let $S$ and $T$ be nonempty sets of real numbers such that every real number is in $S$ or $T$ and if $s \in S$ and $t \in T$, then $s < t$. Prove that there is a unique real number $\beta$ such that every real number less than $\beta$ is in $S$ and every real number greater than $\beta$ is in $T$. (A decomposition of the reals into two sets with these properties is a *Dedekind cut*. This is known as *Dedekind's theorem*.)

**9.** Using properties **(A)**–**(H)** of the real numbers and taking Dedekind's theorem (Exercise 8) as given, show that every nonempty set $U$ of real numbers that is bounded above has a supremum. HINT: *Let $T$ be the set of upper bounds of $U$ and $S$ be the set of real numbers that are not upper bounds of $U$.*

**10.** Let $S$ and $T$ be nonempty sets of real numbers and define

$$S + T = \left\{ s + t \;\middle|\; s \in S, t \in T \right\}.$$

**(a)** Show that

$$\sup(S + T) = \sup S + \sup T \tag{A}$$

if $S$ and $T$ are bounded above and

$$\inf(S + T) = \inf S + \inf T \tag{B}$$

if $S$ and $T$ are bounded below.

**(b)** Show that if they are properly interpreted in the extended reals, then (A) and (B) hold if $S$ and $T$ are arbitrary nonempty sets of real numbers.

**11.** Let $S$ and $T$ be nonempty sets of real numbers and define

$$S - T = \left\{ s - t \;\middle|\; s \in S, t \in T \right\}.$$

**(a)** Show that if $S$ and $T$ are bounded, then

$$\sup(S - T) = \sup S - \inf T \tag{A}$$

and

$$\inf(S - T) = \inf S - \sup T. \tag{B}$$

**(b)** Show that if they are properly interpreted in the extended reals, then (A) and (B) hold if $S$ and $T$ are arbitrary nonempty sets of real numbers.

**12.** Let $S$ be a bounded nonempty set of real numbers, and let $a$ and $b$ be fixed real numbers. Define $T = \left\{ as + b \;\middle|\; s \in S \right\}$. Find formulas for $\sup T$ and $\inf T$ in terms of $\sup S$ and $\inf S$. Prove your formulas.

## 1.2 MATHEMATICAL INDUCTION

If a flight of stairs is designed so that falling off any step inevitably leads to falling off the next, then falling off the first step is a sure way to end up at the bottom. Crudely expressed, this is the essence of the *principle of mathematical induction*: If the truth of a statement depending on a given integer $n$ implies the truth of the corresponding statement with $n$ replaced by $n + 1$, then the statement is true for all positive integers $n$ if it is true for $n = 1$. Although you have probably studied this principle before, it is so important that it merits careful review here.

### Peano's Postulates and Induction

The rigorous construction of the real number system starts with a set $\mathbb{N}$ of undefined elements called *natural numbers*, with the following properties.

**(A)** $\mathbb{N}$ is nonempty.

**(B)** Associated with each natural number $n$ there is a unique natural number $n'$ called the *successor of* $n$.

**(C)** There is a natural number $\overline{n}$ that is not the successor of any natural number.

**(D)** Distinct natural numbers have distinct successors; that is, if $n \neq m$, then $n' \neq m'$.

**(E)** The only subset of $\mathbb{N}$ that contains $\overline{n}$ and the successors of all its elements is $\mathbb{N}$ itself.

These axioms are known as *Peano's postulates*. The real numbers can be constructed from the natural numbers by definitions and arguments based on them. This is a formidable task that we will not undertake. We mention it to show how little you need to start with to construct the reals and, more important, to draw attention to postulate **(E)**, which is the basis for the principle of mathematical induction.

It can be shown that the positive integers form a subset of the reals that satisfies Peano's postulates (with $\overline{n} = 1$ and $n' = n + 1$), and it is customary to regard the positive integers and the natural numbers as identical. From this point of view, the principle of mathematical induction is basically a restatement of postulate **(E)**.

**Theorem 1.2.1 (Principle of Mathematical Induction)** *Let* $P_1$, $P_2$,..., $P_n$,
... *be propositions, one for each positive integer, such that*

**(a)** $P_1$ *is true*;

**(b)** *for each positive integer* $n$, $P_n$ *implies* $P_{n+1}$.

*Then* $P_n$ *is true for each positive integer* $n$.

**Proof** Let

$$\mathbb{M} = \left\{ n \,\middle|\, n \in \mathbb{N} \text{ and } P_n \text{ is true} \right\}.$$

From **(a)**, $1 \in \mathbb{M}$, and from **(b)**, $n + 1 \in \mathbb{M}$ whenever $n \in \mathbb{M}$. Therefore, $\mathbb{M} = \mathbb{N}$, by postulate **(E)**. □

**Example 1.2.1** Let $P_n$ be the proposition that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

Then $P_1$ is the proposition that $1 = 1$, which is certainly true. If $P_n$ is true, then adding $n + 1$ to both sides of (1) yields

$$\begin{aligned}(1 + 2 + \cdots + n) + (n + 1) &= \frac{n(n+1)}{2} + (n + 1)\\ &= (n + 1)\left(\frac{n}{2} + 1\right)\\ &= \frac{(n + 1)(n + 2)}{2},\end{aligned}$$

or

$$1 + 2 + \cdots + (n + 1) = \frac{(n + 1)(n + 2)}{2},$$

which is $P_{n+1}$, since it has the form of (1), with $n$ replaced by $n+1$. Hence, $P_n$ implies $P_{n+1}$, so (1) is true for all $n$, by Theorem 1.2.1. ■

A proof based on Theorem 1.2.1 is an *induction proof*, or *proof by induction*. The assumption that $P_n$ is true is the *induction assumption*. (Theorem 1.2.3 permits a kind of induction proof in which the induction assumption takes a different form.)

Induction, by definition, can be used only to verify results conjectured by other means. Thus, in Example 1.2.1 we did not use induction to *find* the sum

$$s_n = 1 + 2 + \cdots + n; \tag{2}$$

rather, we *verified* that

$$s_n = \frac{n(n+1)}{2}. \tag{3}$$

How you guess what to prove by induction depends on the problem and your approach to it. For example, (3) might be conjectured after observing that

$$s_1 = 1 = \frac{1 \cdot 2}{2}, \quad s_2 = 3 = \frac{2 \cdot 3}{2}, \quad s_3 = 6 = \frac{4 \cdot 3}{2}.$$

However, this requires sufficient insight to recognize that these results are of the form (3) for $n = 1$, 2, and 3. Although it is easy to prove (3) by induction once it has been conjectured, induction is not the most efficient way to find $s_n$, which can be obtained quickly by rewriting (2) as

$$s_n = n + (n-1) + \cdots + 1$$

and adding this to (2) to obtain

$$2s_n = [n+1] + [(n-1)+2] + \cdots + [1+n].$$

There are $n$ bracketed expressions on the right, and the terms in each add up to $n+1$; hence,

$$2s_n = n(n+1),$$

which yields (3).

The next two examples deal with problems for which induction is a natural and efficient method of solution.

**Example 1.2.2** Let $a_1 = 1$ and

$$a_{n+1} = \frac{1}{n+1}a_n, \quad n \ge 1 \tag{4}$$

(we say that $a_n$ is defined *inductively*), and suppose that we wish to find an explicit formula for $a_n$. By considering $n = 1$, 2, and 3, we find that

$$a_1 = \frac{1}{1}, \quad a_2 = \frac{1}{1 \cdot 2}, \quad \text{and} \quad a_3 = \frac{1}{1 \cdot 2 \cdot 3},$$

and therefore we conjecture that

$$a_n = \frac{1}{n!}. \tag{5}$$

This is given for $n = 1$. If we assume it is true for some $n$, substituting it into (4) yields

$$a_{n+1} = \frac{1}{n+1}\frac{1}{n!} = \frac{1}{(n+1)!},$$

which is (5) with $n$ replaced by $n + 1$. Therefore, (5) is true for every positive integer $n$, by Theorem 1.2.1. ■

**Example 1.2.3** For each nonnegative integer $n$, let $x_n$ be a real number and suppose that

$$|x_{n+1} - x_n| \le r|x_n - x_{n-1}|, \quad n \ge 1, \tag{6}$$

where $r$ is a fixed positive number. By considering (6) for $n = 1, 2$, and 3, we find that

$$\begin{aligned}
&|x_2 - x_1| \le r|x_1 - x_0|, \\
&|x_3 - x_2| \le r|x_2 - x_1| \le r^2|x_1 - x_0|, \\
&|x_4 - x_3| \le r|x_3 - x_2| \le r^3|x_1 - x_0|.
\end{aligned}$$

Therefore, we conjecture that

$$|x_n - x_{n-1}| \le r^{n-1}|x_1 - x_0| \quad \text{if} \quad n \ge 1. \tag{7}$$

This is trivial for $n = 1$. If it is true for some $n$, then (6) and (7) imply that

$$|x_{n+1} - x_n| \le r(r^{n-1}|x_1 - x_0|), \quad \text{so} \quad |x_{n+1} - x_n| \le r^n|x_1 - x_0|,$$

which is proposition (7) with $n$ replaced by $n + 1$. Hence, (7) is true for every positive integer $n$, by Theorem 1.2.1. ■

The major effort in an induction proof (after $P_1, P_2, \dots, P_n, \dots$ have been formulated) is usually directed toward showing that $P_n$ implies $P_{n+1}$. However, it is important to verify $P_1$, since $P_n$ may imply $P_{n+1}$ even if some or all of the propositions $P_1, P_2, \dots, P_n, \dots$ are false.

**Example 1.2.4** Let $P_n$ be the proposition that $2n - 1$ is divisible by 2. If $P_n$ is true then $P_{n+1}$ is also, since

$$2n + 1 = (2n - 1) + 2.$$

However, we cannot conclude that $P_n$ is true for $n \ge 1$. In fact, $P_n$ is false for every $n$. ■

The following formulation of the principle of mathematical induction permits us to start induction proofs with an arbitrary integer, rather than 1, as required in Theorem 1.2.1.

**Theorem 1.2.2** *Let $n_0$ be any integer (positive, negative, or zero). Let $P_{n_0}$, $P_{n_0+1}$, $\dots$, $P_n, \dots$ be propositions, one for each integer $n \ge n_0$, such that*

**(a)** *$P_{n_0}$ is true;*

**(b)** *for each integer $n \ge n_0$, $P_n$ implies $P_{n+1}$.*

*Then $P_n$ is true for every integer $n \ge n_0$.*

**Proof** For $m \ge 1$, let $Q_m$ be the proposition defined by $Q_m = P_{m+n_0-1}$. Then $Q_1 = P_{n_0}$ is true by **(a)**. If $m \ge 1$ and $Q_m = P_{m+n_0-1}$ is true, then $Q_{m+1} = P_{m+n_0}$ is true by **(b)** with $n$ replaced by $m + n_0 - 1$. Therefore, $Q_m$ is true for all $m \ge 1$ by Theorem 1.2.1 with $P$ replaced by $Q$ and $n$ replaced by $m$. This is equivalent to the statement that $P_n$ is true for all $n \ge n_0$. $\square$

**Example 1.2.5** Consider the proposition $P_n$ that

$$3n + 16 > 0.$$

If $P_n$ is true, then so is $P_{n+1}$, since

$$\begin{aligned} 3(n+1) + 16 &= 3n + 3 + 16 \\ &= (3n + 16) + 3 > 0 + 3 \text{ (by the induction assumption)} \\ &> 0. \end{aligned}$$

The smallest $n_0$ for which $P_{n_0}$ is true is $n_0 = -5$. Hence, $P_n$ is true for $n \ge -5$, by Theorem 1.2.2. $\blacksquare$

**Example 1.2.6** Let $P_n$ be the proposition that

$$n! - 3^n > 0.$$

If $P_n$ is true, then

$$\begin{aligned} (n+1)! - 3^{n+1} &= n!(n+1) - 3^{n+1} \\ &> 3^n(n+1) - 3^{n+1} \quad \text{(by the induction assumption)} \\ &= 3^n(n-2). \end{aligned}$$

Therefore, $P_n$ implies $P_{n+1}$ if $n > 2$. By trial and error, $n_0 = 7$ is the smallest integer such that $P_{n_0}$ is true; hence, $P_n$ is true for $n \ge 7$, by Theorem 1.2.2. $\blacksquare$

The next theorem is a useful consequence of the principle of mathematical induction.

**Theorem 1.2.3** *Let $n_0$ be any integer (positive, negative, or zero). Let $P_{n_0}$, $P_{n_0+1}, \dots$, $P_n, \dots$ be propositions, one for each integer $n \ge n_0$, such that*

**(a)** *$P_{n_0}$ is true;*

**(b)** *for $n \ge n_0$, $P_{n+1}$ is true if $P_{n_0}$, $P_{n_0+1}, \dots$, $P_n$ are all true.*

*Then $P_n$ is true for $n \ge n_0$.*

**Proof** For $n \geq n_0$, let $Q_n$ be the proposition that $P_{n_0}, P_{n_0+1}, \dots, P_n$ are all true. Then $Q_{n_0}$ is true by **(a)**. Since $Q_n$ implies $P_{n+1}$ by **(b)**, and $Q_{n+1}$ is true if $Q_n$ and $P_n$ are both true, Theorem 1.2.2 implies that $Q_n$ is true for all $n \geq n_0$. Therefore, $P_n$ is true for all $n \geq n_0$. □

**Example 1.2.7** An integer $p > 1$ is a *prime* if it cannot be factored as $p = rs$ where $r$ and $s$ are integers and $1 < r, s < p$. Thus, 2, 3, 5, 7, and 11 are primes, and, although 4, 6, 8, 9, and 10 are not, they are products of primes:

$$4 = 2 \cdot 2, \quad 6 = 2 \cdot 3, \quad 8 = 2 \cdot 2 \cdot 2, \quad 9 = 3 \cdot 3, \quad 10 = 2 \cdot 5.$$

These observations suggest that each integer $n \geq 2$ is a prime or a product of primes. Let this proposition be $P_n$. Then $P_2$ is true, but neither Theorem 1.2.1 nor Theorem 1.2.2 apply, since $P_n$ does not imply $P_{n+1}$ in any obvious way. (For example, it is not evident from $24 = 2 \cdot 2 \cdot 2 \cdot 3$ that 25 is a product of primes.) However, Theorem 1.2.3 yields the stated result, as follows. Suppose that $n \geq 2$ and $P_2, \dots, P_n$ are true. Either $n+1$ is a prime or

$$n + 1 = rs, \tag{8}$$

where $r$ and $s$ are integers and $1 < r, s < n$, so $P_r$ and $P_s$ are true by assumption. Hence, $r$ and $s$ are primes or products of primes and (8) implies that $n+1$ is a product of primes. We have now proved $P_{n+1}$ (that $n+1$ is a prime or a product of primes). Therefore, $P_n$ is true for all $n \geq 2$, by Theorem 1.2.3. ■

## 1.2 Exercises

*Prove the assertions in Exercises* 1–6 *by induction.*

**1.** The sum of the first $n$ odd integers is $n^2$.

**2.** $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

**3.** $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$.

**4.** If $a_1, a_2, \dots, a_n$ are arbitrary real numbers, then

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

**5.** If $a_i \geq 0$, $i \geq 1$, then

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

**6.** If $0 \leq a_i \leq 1$, $i \geq 1$, then

$$(1 - a_1)(1 - a_2) \cdots (1 - a_n) \geq 1 - a_1 - a_2 \cdots - a_n.$$