N. BOURBAKI

ELEMENTS OF MATHEMATICS

Functions of a Real Variable

Elementary Theory

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To the reader

1. The Elements of Mathematics Series takes up mathematics at the beginning, and gives complete proofs. In principle, it requires no particular knowledge of mathematics on the readers' part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought. Nevertheless it is directed especially to those who have a good knowledge of at least the content of the first year or two of a university mathematics course.

2. The method of exposition we have chosen is axiomatic, and normally proceeds from the general to the particular. The demands of proof impose a rigorously fixed order on the subject matter. It follows that the utility of certain considerations may not be immediately apparent to the reader until later chapters unless he has already a fairly extended knowledge of mathematics.

3. The series is divided into Books and each Book into chapters. The Books already published, either in whole or in part, in the French edition, are listed below. When an English translation is available, the corresponding English title is mentioned between parentheses. Throughout the volume a reference indicates the English edition, when available, and the French edition otherwise.

Théorie des Ensembles (Theory of Sets)	designated by	E	(*Set Theory*)
Algèbre (Algebra) [1]	—	A	(*Alg*)
Topologie Générale (General Topology)	—	TG	(*Gen. Top.*)
Fonctions d'une Variable Réelle (Functions of a Real Variable) [2]	—	FVR	(*FRV*)
Espaces Vectoriels Topologiques (Topological Vector Spaces)	—	EVT	(*Top. Vect. Sp.*)
Intégration	—	INT	
Algèbre Commutative (Commutative Algebra) [3]	—	AC	(*Comm. Alg.*)
Variétes Différentielles et Analytiques	—	VAR	
Groupes et Algèbres de Lie (Lie Groups and Lie Algebras) [4]	—	LIE	(LIE)
Théories Spectrales		TS	

In the first six Books (according to the above order), every statement in the text assumes as known only those results which have already discussed in the same

[1] So far, chapters I to VII only have been translated.
[2] This volume!
[3] So far, chapters I to VII only have been translated.
[4] So far, chapters I to III only have been translated.

chapter, or in the previous chapters ordered as follows: E ; A, chapters I to III ; TG, chapters I to III ; A, from chapter IV on ; TG, from chapter IV on ; FVR ; EVT ; INT.

From the seventh Book on, the reader will usually find a precise indication of its logical relationship to the other Books (the first six Books being always assumed to be known).

4. However, we have sometimes inserted examples in the text which refer to facts which the reader may already know but which have not yet been discussed in the Series. Such examples are placed between two asterisks : *. . . *. Most readers will undoubtedly find that these examples will help them to understand the text. In other cases, the passages between *. . . * refer to results which are discussed elsewhere in the Series. We hope the reader will be able to verify the absence of any vicious circle.

5. The logical framework of each chapter consists of the *definitions,* the *axioms,* and the *theorems* of the chapter. These are the parts that have mainly to be borne in mind for subsequent use. Less important results and those which can easily be deduced from the theorems are labelled as "propositions", "lemmas", "corollaries", "remarks", etc. Those which may be omitted at a first reading are printed in small type. A commentary on a particularly important theorem appears occasionally under the name of "scholium".

To avoid tedious repetitions it is sometimes convenient to introduce notation or abbreviations which are in force only within a certain chapter or a certain section of a chapter (for example, in a chapter which is concerned only with commutative rings, the word "ring" would always signify "commutative ring"). Such conventions are always explicitly mentioned, generally at the beginning of the *chapter* in which they occur.

6. Some passages are designed to forewarn the reader against serious errors. These passages are signposted in the margin with the sign (“dangerous bend”).

7. The Exercises are designed both to enable the reader to satisfy himself that he has digested the text and to bring to his notice results which have no place in the text but which are nonetheless of interest. The most difficult exercises bear the sign ¶.

8. In general we have adhered to the commonly accepted terminology, *except where there appeared to be good reasons for deviating from it.*

9. We have made a particular effort always to use rigorously correct language, without sacrificing simplicity. As far as possible we have drawn attention in the text to *abuses of language,* without which any mathematical text runs the risk of pedantry, not to say unreadability.

10. Since in principle the text consists of a dogmatic exposition of a theory, it contains in general no references to the literature. Bibliographical are gathered together in *Historical Notes*. The bibliography which follows each historical note contains in general only those books and original memoirs which have been of the greatest importance in the evolution of the theory under discussion. It makes no sort of pretence to completeness.

As to the exercises, we have not thought it worthwhile in general to indicate their origins, since they have been taken from many different sources (original papers, textbooks, collections of exercises).

11. In the present Book, references to theorems, axioms, definitions, . . . are given by quoting successively:

– the Book (using the abbreviation listed in Section 3), chapter and page, where they can be found ;

– the chapter and page only when referring to the present Book.

The *Summaries of Results* are quoted by to the letter R; thus *Set Theory,* R signifies "*Summary of Results of the Theory of Sets*".

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CHAPTER V LOCAL STUDY OF FUNCTIONS

CHAPTER VII THE GAMMA FUNCTION

INTRODUCTION

The purpose of this Book is the elementary study of the infinitesimal properties of *one* real variable; the extension of these properties to functions of *several* real variables, or, all the more, to functions defined on more general spaces, will be treated only in later Books.

The results which we shall demonstrate will be useful above all in relation to (finite) real-valued functions of a real variable; but most of them extend without further argument to functions of a real variable taking values in a *topological vector space* over **R** (see below); as these functions occur frequently in Analysis we shall state for them all results which are not specific to real-valued functions.

The notion of a topological vector space, of which we have just spoken, is defined and studied in detail in Book V of this Series; but we do not need *any* of the results of Book V in this Book; some definitions, however, are needed, and we shall reproduce them below for the convenience of the reader.

We shall not repeat the definition of a *vector space* over a (*commutative*) *field* K (*Alg*., II, p. 193). [1] A *topological vector space* E over a *topological field* K is a vector space over K endowed with a topology such that the functions $\mathbf{x} + \mathbf{y}$ and $\mathbf{x}t$ are *continuous* on $E \times E$ and $E \times K$ respectively; in particular, such a topology is compatible with the structure of the additive group of E. All topological vector spaces considered in this Book are implicitly assumed to be Hausdorff. When the topological group E is complete one says that the topological vector space E is *complete*. Every *normed* vector space over a *valued field* K (*Gen. Top*., IX, p. 169) [2] is a topological vector space over K.

Let E be a vector space (with or without a topology) over the real field **R**; if **x**, **y** are arbitrary points in E the set of points $\mathbf{x}t + \mathbf{y}(1-t)$ where t runs through the closed

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[1] The elements (or *vectors*) of a vector space E over a commutative field K will usually be denoted in this chapter by thick minuscules, and scalars by roman minuscules; most often we shall place the scalar t to the *right* in the product of a vector **x** by t, writing the product as $\mathbf{x}t$; on occasion we will allow ourselves to use the left notation $t\mathbf{x}$ in certain cases where it is more convenient; also, sometimes we shall write the product of the scalar $1/t$ ($t \neq 0$) and the vector **x** in the form $\mathbf{x}/t$.

[2] We recall that a *norm* on E is a real function $\|\mathbf{x}\|$ defined on E, taking finite non-negative values, such that the relation $\|\mathbf{x}\| = 0$ is equivalent to $\mathbf{x} = 0$ and such that

$$\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\| \quad \text{and} \quad \|\mathbf{x}t\| = \|\mathbf{x}\| \,.\, |t|$$

for all $t \in K$ ($|t|$ being the absolute value of t in K).

segment $[0,\ 1]$ of $\mathbf{R}$ is called the *closed segment* with endpoints $\mathbf{x}$, $\mathbf{y}$. One says that a subset A of E is *convex* if for any $\mathbf{x}$, $\mathbf{y}$ in A the closed segment with endpoints $\mathbf{x}$ and $\mathbf{y}$ is contained in A. For example, an affine linear variety is convex; so is any closed segment; in $\mathbf{R}^n$ any parallelotope (*Gen. Top.*, VI, p. 34) is convex. Every intersection of convex sets is convex.

We say that a topological vector space E over the field $\mathbf{R}$ is *locally convex* if the origin (and thus any point of E) has a fundamental system of *convex* neighbourhoods. Every *normed* space is locally convex; indeed, the balls $\|\mathbf{x}\| \leqslant r$ $(r > 0)$ form a fundamental system of neighbourhoods of 0 in E, and each of these is convex, for the relations $\|\mathbf{x}\| \leqslant r$, $\|\mathbf{y}\| \leqslant r$ imply that

$$\|\mathbf{x}t + \mathbf{y}(1 - t)\| \leqslant \|\mathbf{x}\|\, t + \|\mathbf{y}\|\, (1 - t) \leqslant r$$

for $0 \leqslant t \leqslant 1$.

Finally, a *topological algebra* A over a (commutative) *topological field* K is an algebra over K endowed with a topology for which the functions $\mathbf{x}+\mathbf{y}$, $\mathbf{xy}$ and $\mathbf{x}t$ are continuous on $A \times A$, $A \times A$ and $A \times K$ respectively; when one endows A only with its topology and vector space structure over K then A is a topological vector space. Every *normed algebra* over a *valued field* K (*Gen. Top.*, IX, p. 175) is a topological algebra over K.

CHAPTER I
Derivatives

§ 1. FIRST DERIVATIVE

As was said in the Introduction, in this chapter and the next we shall study the infinitesimal properties of functions which are defined on a subset of the real field **R** and take their values in a *Hausdorff topological vector space* E over the field **R**; for brevity we shall say that such a function is a *vector function of a real variable*. The most important case is that where $E = \mathbf{R}$ (real-valued functions of a real variable). When $E = \mathbf{R}^n$, consideration of a vector function with values in E reduces to the simultaneous consideration of n finite real functions.

Many of the definitions and properties stated in chapter I extend to functions which are defined on a subset of the field **C** of complex numbers and take their values in a topological vector space over **C** (vector functions of a complex variable). Some of these definitions and properties extend even to functions which are defined on a subset of an arbitrary commutative *topological field* K and take their values in a topological vector space over K.

We shall indicate these generalizations in passing (see in particular I, p. 10, *Remark 2*), emphasising above all the case of functions of a complex variable, which are by far the most important, together with functions of a real variable, and will be studied in greater depth in a later Book.

1. DERIVATIVE OF A VECTOR FUNCTION

DEFINITION 1. *Let* $\mathbf{f}$ *be a vector function defined on an interval* $I \subset \mathbf{R}$ *which does not reduce to a single point. We say that* $\mathbf{f}$ *is differentiable at a point* $x_0 \in I$ *if* $\lim\limits_{x \to x_0, x \in I, x \neq x_0} \dfrac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}$ *exists* (*in the vector space where* $\mathbf{f}$ *takes its values*)*; the value of this limit is called the first derivative* (or simply the *derivative*) *of* $\mathbf{f}$ *at the point* x_0, *and it is denoted by* $\mathbf{f}'(x_0)$ *or* $\mathrm{D}\mathbf{f}(x_0)$.

If $\mathbf{f}$ is differentiable at the point x_0, so is the *restriction* of $\mathbf{f}$ to any interval $J \subset I$ which does not reduce to a single point and such that $x_0 \in J$; and the derivative of this restriction is equal to $\mathbf{f}'(x_0)$. Conversely, let J be an interval contained in I and containing a neighbourhood of x_0 relative to I; if the restriction of $\mathbf{f}$ to J admits a derivative at the point x_0, then so does $\mathbf{f}$.

We summarise these properties by saying that the concept of derivative is a *local* concept.

Remarks. *1) In Kinematics, if the point $\mathbf{f}(t)$ is the position of a moving point in the space $\mathbf{R}^3$ at time t, then $\frac{\mathbf{f}(t)-\mathbf{f}(t_0)}{t-t_0}$ is termed the *average velocity* between the instants t_0 and t, and its limit $\mathbf{f}'(t_0)$ is the *instantaneous velocity* (or simply *velocity*) at the time t_0 (when this limit exists).*

2) If a function $\mathbf{f}$, defined on I, is differentiable at a point $x_0 \in \mathrm{I}$, it is necessarily *continuous relative to* I at this point.

DEFINITION 2. *Let* $\mathbf{f}$ *be a vector function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *and let* x_0 *be a point of* I *such that the interval* $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap\,]-\infty, x_0]$) *does not reduce to a single point. We say that* $\mathbf{f}$ *is differentiable on the right* (resp. *on the left*) *at the point* x_0 *if the restriction of* $\mathbf{f}$ *to the interval* $\mathrm{I} \cap [x_0, +\infty[$ (resp. $\mathrm{I} \cap\,]-\infty, x_0]$) *is differentiable at the point* x_0; *the value of the derivative of this restriction at the point* x_0 *is called the right* (resp. *left*) *derivative of* $\mathbf{f}$ *at the point* x_0 *and is denoted by* $\mathbf{f}'_d(x_0)$ (resp. $\mathbf{f}'_g(x_0)$).

Let $\mathbf{f}$ be a vector function defined on I, and x_0 an *interior* point of I such that $\mathbf{f}$ is continuous at this point; it follows from defs. 1 and 2 that for $\mathbf{f}$ to be differentiable at x_0 it is necessary and sufficient that $\mathbf{f}$ admit both a right and a left derivative at this point, and that these derivatives be *equal;* and then

$$\mathbf{f}'(x_0) = \mathbf{f}'_d(x_0) = \mathbf{f}'_g(x_0).$$

Examples. 1) A *constant* function has zero derivative at every point.

2) An affine linear function $x \mapsto \mathbf{a}x + \mathbf{b}$ has derivative equal to $\mathbf{a}$ at every point.

3) The real function $1/x$ (defined for $x \neq 0$) is differentiable at each point $x_0 \neq 0$, for we have $\left(\frac{1}{x} - \frac{1}{x_0}\right) \Big/ (x - x_0) = -\frac{1}{xx_0}$, and, since $1/x$ is continuous at x_0, the limit of the preceding expression is $-1/x_0^2$.

4) The scalar function $|x|$, defined on $\mathbf{R}$, has right derivative $+1$ and left derivative -1 at $x = 0$; it is not differentiable at this point.

5) The real function equal to 0 for $x = 0$, and to $x \sin 1/x$ for $x \neq 0$, is defined and continuous on $\mathbf{R}$, but has neither right nor left derivative at the point $x \neq 0$. One can give examples of functions which are continuous on an interval and fail to have a derivative at *every* point of the interval (I, p. 35, exerc. 2 and 3).

DEFINITION 3. *We say that a vector function* $\mathbf{f}$ *defined on an interval* $\mathrm{I} \subset \mathbf{R}$ *is differentiable* (resp. *right differentiable, left differentiable*) *on* I *if it is differentiable* (resp. *right differentiable, left differentiable*) *at each point of* I; *the function* $x \mapsto \mathbf{f}'(x)$ (resp. $x \mapsto \mathbf{f}'_d(x)$, $x \mapsto \mathbf{f}'_g(x)$) *defined on* I, *is called the derived function, or* (by abuse of language) *the derivative* (resp. *right derivative, left derivative*) *of* $\mathbf{f}$, *and is denoted by* $\mathbf{f}'$ *or* $\mathrm{D}\mathbf{f}$ *or* $d\mathbf{f}/dx$ (resp. $\mathbf{f}'_d$, $\mathbf{f}'_g$).

Remark. A function may be differentiable on an interval without its derivative being continuous at every point of the interval (*cf.* I, p. 36, exerc. 5); *this is shown by the example of the function equal to 0 for $x = 0$ and to $x^2 \sin 1/x$ for $x \neq 0$; it has a derivative everywhere, but this derivative is discontinuous at the point $x = 0$.$_*$

2. LINEARITY OF DIFFERENTIATION

PROPOSITION 1. *The set of vector functions defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in a given topological vector space* E, *and differentiable at the point* x_0, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}(x_0)$ *is a linear mapping of this space into* E.

In other words, if $\mathbf{f}$ and $\mathbf{g}$ are defined on I and differentiable at the point x_0, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f}a$ (a an arbitrary scalar) are differentiable at x_0 and their derivatives there are $\mathbf{f}'(x_0) + \mathbf{g}'(x_0)$ and $\mathbf{f}'(x_0)a$ respectively. This follows immediately from the continuity of $\mathbf{x} + \mathbf{y}$ and of $\mathbf{x}a$ on $\mathrm{E} \times \mathrm{E}$ and E respectively.

COROLLARY. *The set of vector functions defined on an interval* I, *taking values in a given topological vector space* E, *and differentiable on* I, *is a vector space over* $\mathbf{R}$, *and the map* $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}$ *is a linear mapping of this space into the vector space of mappings from* I *into* E.

Remark. If one endows the vector space of mappings from I into E and its subspace of differentiable mappings (*cf. Gen. Top.*, X, p. 277) with the topology of simple convergence (or the topology of uniform convergence), the linear mapping $\mathbf{f} \mapsto \mathrm{D}\mathbf{f}$ *is not continuous* (in general) *for example, the sequence of functions $\mathbf{f}_n(x) = \sin n^2x/n$ converges uniformly to 0 on $\mathbf{R}$, but the sequence of derivatives $\mathbf{f}_n'(x) = n \cos n^2x$ does not converge even simply to 0.$_*$

PROPOSITION 2. *Let* E *and* F *be two topological vector spaces over* $\mathbf{R}$, *and* $\mathbf{u}$ *a continuous linear map from* E *into* F. *If* $\mathbf{f}$ *is a vector function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *taking values in* E, *and differentiable at the point* $x_0 \in \mathrm{I}$, *then the composite function* $\mathbf{u} \circ \mathbf{f}$ *has a derivative equal to* $\mathbf{u}(\mathbf{f}'(x_0))$ *at* x_0.

Indeed, since $\dfrac{\mathbf{u}(\mathbf{f}(x)) - \mathbf{u}(\mathbf{f}(x_0))}{x - x_0} = \mathbf{u}\left(\dfrac{\mathbf{f}(x) - \mathbf{f}(x_0)}{x - x_0}\right)$, this follows from the continuity of $\mathbf{u}$.

COROLLARY. *If* φ *is a continuous linear form on* E, *then the real function* $\varphi \circ \mathbf{f}$ *has a derivative equal to* $\varphi(\mathbf{f}'(x_0))$ *at the point* x_0.

Examples. 1) Let $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ be a function with values in $\mathbf{R}^n$, defined on an interval $\mathrm{I} \subset \mathbf{R}$; each real function f_i is none other than the composite function $\mathrm{pr}_i \circ \mathbf{f}$, so is differentiable at the point x_0 if $\mathbf{f}$ is, and, if so, $\mathbf{f}'(x_0) = (f_i'(x_0))_{1 \leqslant i \leqslant n}$.

*2) In Kinematics, if $\mathbf{f}(t)$ is the position of a moving point M at time t, if $\mathbf{g}(t)$ is the position at the same instant of the projection M′ of M onto a plane P (resp. a line D) with kernel a line (resp. a plane) not parallel to P (resp. D), then $\mathbf{g}$ is the composition of the projection $\mathbf{u}$ of $\mathbf{R}^3$ onto P (resp. D) and of $\mathbf{f}$; since $\mathbf{u}$ is a (continuous) linear mapping one sees that the projection of the velocity of a moving point onto a plane (resp. a line) is equal to the velocity of the projection of the moving point onto the plane (resp. line).$_*$

3) Let f be a complex-valued function defined on an interval $\mathrm{I} \subset \mathbf{R}$, and let a be an arbitrary complex number; prop. 2 shows that if f is differentiable at a point x_0 then so is af, and the derivative of this function at x_0 is equal to $af'(x_0)$.

3. DERIVATIVE OF A PRODUCT

Let us now consider p topological vector spaces E_i $(1 \leqslant i \leqslant p)$ over $\mathbf{R}$, and a continuous multilinear [1] map (which we shall denote by

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p])$$

of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into a topological vector space F over $\mathbf{R}$.

PROPOSITION 3 . *For each index i $(1 \leqslant i \leqslant p)$ let $\mathbf{f}_i$ be a function defined on an interval $\mathrm{I} \subset \mathbf{R}$, taking values in E_i, and differentiable at the point $x_0 \in \mathrm{I}$. Then the function*

$$x \mapsto [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$$

defined on I *with values in* F *has a derivative equal to*

$$\sum_{i=1}^{p} \left[\mathbf{f}_1(x_0) \ldots \mathbf{f}_{i-1}(x_0).\mathbf{f}_i'(x_0).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0)\right] \tag{1}$$

at x_0.

Let us put $\mathbf{h}(x) = [\mathbf{f}_1(x).\mathbf{f}_2(x) \ldots \mathbf{f}_p(x)]$; then, by the identity

$$\left[\mathbf{b}_1.\mathbf{b}_2 \ldots \mathbf{b}_p\right] - \left[\mathbf{a}_1.\mathbf{a}_2 \ldots \mathbf{a}_p\right] = \sum_{i=1}^{p} \left[\mathbf{b}_1 \ldots \mathbf{b}_{i-1}.(\mathbf{b}_i - \mathbf{a}_i).\mathbf{a}_{i+1} \ldots \mathbf{a}_p\right],$$

we can write

$$\mathbf{h}(x) - \mathbf{h}(x_0) = \sum_{i=1}^{p} \left[\mathbf{f}_1(x) \ldots \mathbf{f}_{i-1}(x).(\mathbf{f}_i(x) - \mathbf{f}_i(x_0)).\mathbf{f}_{i+1}(x_0) \ldots \mathbf{f}_p(x_0)\right].$$

On multiplying both sides by $\dfrac{1}{x - x_0}$ and letting x approach x_0 in I, we obtain the expression (1), since both the map

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p) \mapsto [\mathbf{x}_1.\mathbf{x}_2 \ldots \mathbf{x}_p]$$

and addition in F are continuous.

[1] Recall (*Alg.*, II, p. 265) that a map $\mathbf{f}$ of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into F is said to be *multilinear* if each partial mapping

$$\mathbf{x}_i \mapsto \mathbf{f}(\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_p)$$

from E_i into F $(1 \leqslant i \leqslant p)$ is a *linear* map, the $\mathbf{a}_j$ for indices $j \neq i$ being arbitrary in E_j. We note that if the E_i are *finite* dimensional over $\mathbf{R}$ then every multilinear map of $\mathrm{E}_1 \times \mathrm{E}_2 \times \cdots \times \mathrm{E}_p$ into F is necessarily *continuous;* this need not be so if some of these spaces are topological vector spaces of infinite dimension.

When some of the functions $\mathbf{f}_i$ are *constant,* the terms in the expression (1) containing their derivatives $\mathbf{f}'_i(x_0)$ are zero.

Let us consider in detail the particular case $p = 2$, the most important in applications: if $(\mathbf{x}, \mathbf{y}) \mapsto [\mathbf{x}.\mathbf{y}]$ is a *continuous bilinear* map of E × F into G, (E, F, G being topological vector spaces over $\mathbf{R}$), and $\mathbf{f}$ and $\mathbf{g}$ are two vector functions, differentiable at x_0, with values in E and F respectively, then the vector function $x \mapsto [\mathbf{f}(x).\mathbf{g}(x)]$ (which we denote by $[\mathbf{f}.\mathbf{g}]$) has a derivative equal to $[\mathbf{f}'(x_0).\mathbf{g}(x_0)] + [\mathbf{f}(x_0).\mathbf{g}'(x_0)]$ at x_0. In particular, if $\mathbf{a}$ is a constant vector, then $[\mathbf{a}.\mathbf{f}]$ (resp. $[\mathbf{f}.\mathbf{a}]$) has a derivative equal to $[\mathbf{a}.\mathbf{f}'(x_0)]$ (resp. $[\mathbf{f}'(x_0).\mathbf{a}]$) at x_0.

If $\mathbf{f}$ and $\mathbf{g}$ are both differentiable on I then so is $[\mathbf{f}.\mathbf{g}]$, and we have

$$[\mathbf{f}.\mathbf{g}]' = [\mathbf{f}'.\mathbf{g}] + [\mathbf{f}.\mathbf{g}']. \tag{2}$$

Examples. 1) Let f be a real function, $\mathbf{g}$ a vector function, both differentiable at a point x_0; the function $\mathbf{g}f$ has a derivative equal to $\mathbf{g}'(x_0)f(x_0) + \mathbf{g}(x_0)f'(x_0)$ at x_0. In particular, if $\mathbf{a}$ is constant, then $\mathbf{a}f$ has derivative $\mathbf{a}f'(x_0)$. This last remark, in conjunction with example 1 of I, p. 5, proves that if $\mathbf{f} = (f_i)_{1 \leqslant i \leqslant n}$ is a vector function with values in $\mathbf{R}^n$, then for $\mathbf{f}$ to be differentiable at the point x_0 it is necessary and sufficient that each of the real functions f_i $(1 \leqslant i \leqslant n)$ be differentiable there: for, if $(\mathbf{e}_i)_{1 \leqslant i \leqslant n}$ is the canonical basis of $\mathbf{R}^n$, we can write $\mathbf{f} = \sum_{i=1}^{n} \mathbf{e}_i f_i$.

2) The real function x^n arises from the multilinear function

$$(x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$$

defined on $\mathbf{R}^n$, by substituting x for each of the x_i; so prop. 3 shows that x^n is differentiable on $\mathbf{R}$ and has derivative nx^{n-1}. As a result the polynomial function $\mathbf{a}_0 x^n + \mathbf{a}_1 x^{n-1} + \cdots + \mathbf{a}_{n-1}x + \mathbf{a}_n$ (the $\mathbf{a}_i$ being constant vectors) has derivative

$$n\mathbf{a}_0 x^{n-1} + (n-1)\mathbf{a}_1 x^{n-2} + \cdots + \mathbf{a}_{n-1};$$

when the $\mathbf{a}_i$ are real numbers this function coincides with the derivative of a polynomial function as defined in Algebra (A, IV).

3) The euclidean *scalar product* $(\mathbf{x} \,|\, \mathbf{y})$ (*Gen. Top.*, VI, p. 40) is a bilinear map (necessarily continuous) of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}$. If $\mathbf{f}$ and $\mathbf{g}$ are two vector functions with values in $\mathbf{R}^n$, and differentiable at the point x_0, then the real function $x \mapsto (\mathbf{f}(x) \,|\, \mathbf{g}(x))$ has a derivative equal to $(\mathbf{f}'(x_0) \,|\, \mathbf{g}(x_0)) + (\mathbf{f}(x_0) \,|\, \mathbf{g}'(x_0))$ at the point x_0. There is an analogous result for the hermitian scalar product on $\mathbf{C}^n$, this space being considered as a vector space *over* $\mathbf{R}$.

Let us consider in particular the case where the euclidean norm $\|\mathbf{f}(x)\|$ is *constant*, so that $(\mathbf{f}(x) \,|\, \mathbf{f}(x)) = \|\mathbf{f}(x)\|^2$ is also constant; on writing that the derivative of $(\mathbf{f}(x) \,|\, \mathbf{f}(x))$ vanishes at x_0 we obtain $(\mathbf{f}(x_0) \,|\, \mathbf{f}'(x_0)) = 0$; in other words, $\mathbf{f}'(x_0)$ is *orthogonal* to $\mathbf{f}(x_0)$.

4) If E is a *topological algebra* over $\mathbf{R}$ (*cf.* Introduction), the product $\mathbf{xy}$ of two elements of E is a continuous bilinear function of $(\mathbf{x}, \mathbf{y})$; if $\mathbf{f}$ and $\mathbf{g}$ have their values in E and are differentiable at the point x_0, then the function $x \mapsto \mathbf{f}(x)\mathbf{g}(x)$ has a derivative equal to $\mathbf{f}'(x_0)\mathbf{g}(x_0) + \mathbf{f}(x_0)\mathbf{g}'(x_0)$ at x_0. In particular, if $U(x) = (\alpha_{ij}(x))$ and $V(x) = (\beta_{ij}(x))$ are two *square matrices* of order n, differentiable at x_0, their product UV has a derivative equal to $U'(x_0)V(x_0) + U(x_0)V'(x_0)$ at x_0 (where $U'(x) = (\alpha'_{ij}(x))$ and $V'(x) = (\beta'_{ij}(x))$).

5) The *determinant* $\det(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ of n vectors $\mathbf{x}_i = (x_{ij})_{1 \leqslant j \leqslant n}$ from the space $\mathbf{R}^n$ (*Alg.*, III, p. 522) being a (continuous) multilinear function of the $\mathbf{x}_i$, one sees that if the

n^2 real functions f_{ij} are differentiable at x_0, then their determinant $g(x) = \det(f_{ij}(x))$ has a derivative equal to

$$\sum_{i=1}^{n} \Big[\mathbf{f}_1(x_0), \ldots, \mathbf{f}_{i-1}(x_0), \mathbf{f}'_i(x_0), \mathbf{f}_{i+1}(x_0), \ldots, \mathbf{f}_n(x_0)\Big]$$

at x_0, where $\mathbf{f}_i(x) = (f_{ij}(x))_{1 \leqslant j \leqslant n}$; in other words, one obtains the derivative of a determinant of order n by taking the sum of the n determinants formed by replacing, for each i, the terms of the i^{th} column by their derivatives.

Remark. If $U(x)$ is a square matrix which is differentiable and invertible at the point x_0, then the derivative of its determinant $\Delta(x) = \det(U(x))$ can be expressed through the derivative of $U(x)$ by the formula

$$\Delta'(x_0) = \Delta(x_0).\mathrm{Tr}(U'(x_0)U^{-1}(x_0)). \tag{3}$$

Indeed, let us put $U(x_0+h) = U(x_0)+hV$; then, by definition, V tends to $U'(x_0)$ when h tends to 0. One can write

$$\Delta(x_0 + h) = \Delta(x_0).\det(I + hVU^{-1}(x_0)).$$

Now $\det(I + hX) = 1 + h\mathrm{Tr}(X) + \sum_{k=2}^{n} \lambda_k h^k$, the λ_k ($k \geqslant 2$) being polynomials in the elements of the matrix X; since the elements of $VU^{-1}(x_0)$ have a limit when h tends to 0, we indeed obtain the formula (3).

4. DERIVATIVE OF THE INVERSE OF A FUNCTION

PROPOSITION 4. *Let* E *be a complete normed algebra with a unit element over* **R** *and let* **f** *be a function defined on an interval* I $\subset$ **R**, *taking values in* E, *and differentiable at the point* $x_0 \in$ I. *If* $\mathbf{y}_0 = \mathbf{f}(x_0)$ *is invertible* [2] *in* E, *then the function* $x \mapsto (\mathbf{f}(x))^{-1}$ *is defined on a neighbourhood of* x_0 *(relative to* I), *and has a derivative equal to* $-\big(\mathbf{f}(x_0)\big)^{-1}\mathbf{f}'(x_0)\big(\mathbf{f}(x_0)\big)^{-1}$ *at* x_0.

Indeed, the set of invertible elements in E is an open set on which the function $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is continuous (*Gen. Top.*, IX, p. 178); since **f** is continuous (relative to I) at x_0, $\big(\mathbf{f}(x)\big)^{-1}$ is defined on a neighbourhood of x_0, and we have

$$\big(\mathbf{f}(x)\big)^{-1} - \big(\mathbf{f}(x_0)\big)^{-1} = \big(\mathbf{f}(x)\big)^{-1}\Big(\mathbf{f}(x_0) - \mathbf{f}(x)\Big)\big(\mathbf{f}(x_0)\big)^{-1}.$$

The proposition thus follows from the continuity of $\mathbf{y}^{-1}$ on a neighbourhood of $\mathbf{y}_0$ and the continuity of $\mathbf{xy}$ on E $\times$ E.

[2] Recall from (*Alg.*, I, p. 15) that an element $\mathbf{z} \in$ E is said to be *invertible* if there exists an element of E, denoted by $\mathbf{z}^{-1}$, such that $\mathbf{z}\mathbf{z}^{-1} = \mathbf{z}^{-1}\mathbf{z} = \mathbf{e}$ (**e** being the unit element of E).

Examples. 1) The most important particular case is that where E is one of the fields **R** or **C** : if f is a function with real or complex values, differentiable at the point x_0, and such that $f(x_0) \neq 0$, then $1/f$ has derivative equal to $-f'(x_0)/(f(x_0))^2$ at x_0.

2) If $U = (\alpha_{ij}(x))$ is a square matrix of order n, differentiable at x_0 and invertible at this point, then U^{-1} has derivative equal to $-U^{-1}U'U^{-1}$ at x_0.

5. DERIVATIVE OF A COMPOSITE FUNCTION

PROPOSITION 5. *Let f be a real function defined on an interval* $\mathrm{I} \subset \mathbf{R}$, *and* $\mathbf{g}$ *a vector function defined on an interval of* **R** *containing* $f(\mathrm{I})$. *If f is differentiable at the point x_0 and* $\mathbf{g}$ *is differentiable at the point $f(x_0)$ then the composite function* $\mathbf{g} \circ f$ *has a derivative equal to* $\mathbf{g}'(f(x_0))f'(x_0)$ *at* x_0.

Let us put $\mathbf{h} = \mathbf{g} \circ f$; for $x \neq x_0$ we can write

$$\frac{\mathbf{h}(x) - \mathbf{h}(x_0)}{x - x_0} = \mathbf{u}(x)\,\frac{f(x) - f(x_0)}{x - x_0}$$

where we set $\mathbf{u}(x) = \dfrac{\mathbf{g}(f(x)) - \mathbf{g}(f(x_0))}{f(x) - f(x_0)}$ if $f(x) \neq f(x_0)$, and $\mathbf{u}(x) = \mathbf{g}'(f(x_0))$ otherwise. Now $f(x)$ has limit $f(x_0)$ when x tends to x_0, so $\mathbf{u}(x)$ has limit $\mathbf{g}'(f(x_0))$, from which the proposition follows in view of the continuity of the function $\mathbf{y}x$ on $\mathrm{E} \times \mathbf{R}$.

6. DERIVATIVE OF AN INVERSE FUNCTION

PROPOSITION 6. *Let f be a homeomorphism of an interval* $\mathrm{I} \subset \mathbf{R}$ *onto an interval* $\mathrm{J} = f(\mathrm{I}) \subset \mathbf{R}$, *and let g be the inverse homeomorphism*[3]. *If f is differentiable at the point* $x_0 \in \mathrm{I}$, *and if* $f'(x_0) \neq 0$, *then g has a derivative equal to* $1/f'(x_0)$ *at* $y_0 = f(x_0)$.

For each $y \in \mathrm{J}$ we have $g(y) \in \mathrm{I}$ and $u = f(g(y))$; we thus can write $\dfrac{g(y) - g(y_0)}{y - y_0} = \dfrac{g(y) - x_0}{f(g(y)) - f(x_0)}$, for $y \neq y_0$. When y tends to y_0 while remaining in J and $\neq y_0$, then $g(y)$ tends to x_0 remaining in I and $\neq x_0$, and the right-hand side in the preceding formula thus has limit $1/f'(x_0)$, since by hypothesis $f'(x_0) \neq 0$.

COROLLARY. *If f is differentiable on* I *and if* $f'(x) \neq 0$ *on* I, *then g is differentiable on* J *and its derivative at each point* $y \in \mathrm{J}$ *is* $1/f'(g(y))$.

For example, for each integer $n > 0$, the function $x^{1/n}$ is a homeomorphism of $\mathbf{R}_+$ onto itself, is the inverse of x^n, and has derivative $\frac{1}{n}x^{\frac{1}{n}-1}$ at each $x > 0$.

One deduces easily, from prop. 5, that for every rational number $r = p/q > 0$ the function $x^r = \left(x^{1/q}\right)^p$ has derivative rx^{r-1} at every $x > 0$.

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[3] For f to be a homeomorphism of I onto a subset of **R** we know that it is necessary and sufficient that f be continuous and strictly monotone on I (*Gen. Top.*, IV, p. 338, th. 5).

Remarks. 1) All the preceding propositions, stated for functions differentiable at a point x_0, immediately yield propositions for functions which are right (resp. left) differentiable at x_0, when, instead of the functions themselves, one considers their restrictions to the intersection of their intervals of definition with the interval $[x_0, +\infty[$ (resp. $]-\infty, x_0]$); we leave it to the reader to state them.

2) The preceding definitions and propositions (except for those concerning right and left derivatives) extend easily to the case where one replaces **R** by an arbitrary *commutative non-discrete topological field* K, and the topological vector spaces (resp. topological algebras) over **R** by topological vector spaces (resp. topological algebras) over K. In def. 1 and props. 1, 2 and 3 it is enough to replace I by a *neighbourhood* of x_0 in K; in prop. 4 one must assume further that the map $\mathbf{y} \mapsto \mathbf{y}^{-1}$ is defined and continuous on a neighbourhood of $\mathbf{f}(x_0)$ in E. Prop. 5 generalizes in the following manner: let K′ be a non-discrete subfield of the topological field K, let E be a topological vector space *over* K; let f be a function defined on a neighbourhood $V \subset \mathrm{K}'$ of $x_0 \in \mathrm{K}'$, with values in K (considered as a topological vector space over K′), differentiable at x_0, and let **g** be a function defined on a neighbourhood of $f(x_0) \in \mathrm{K}$, with values in E, and differentiable at the point $f(x_0)$; then the map $\mathbf{g} \circ f$ is differentiable at x_0 and has derivative $\mathbf{g}'(f(x_0))f'(x_0)$ there (E being then considered as a topological vector space *over* K′).

With the same notation, let **f** be a function defined on a neighbourhood V of $a \in \mathrm{K}$, with values in E, and differentiable at the point a; if $a \in \mathrm{K}'$, then the *restriction* of **f** to $\mathrm{V} \cap \mathrm{K}'$ is differentiable at a, and has derivative $\mathbf{f}'(a)$ there. These considerations apply above all, in practice, to the case where $\mathrm{K} = \mathbf{C}$ and $\mathrm{K}' = \mathbf{R}$.

Finally, prop. 6 extends to the case where one replaces I by a neighbourhood of $x_0 \in \mathrm{K}$, and f by a homeomorphism of I onto a neighbourhood $\mathrm{J} = f(\mathrm{I})$ of $y_0 = f(x_0)$ in K.

7. DERIVATIVES OF REAL-VALUED FUNCTIONS

The preceding definitions and propositions may be augmented in several respects when we deal with *real-valued* functions of a real variable.

In the first place, if f is such a function, defined on an interval $\mathrm{I} \subset \mathbf{R}$, and continuous relative to I at a point $x_0 \in \mathrm{I}$, it can happen that when x tends to x_0 while remaining in I and $\neq x_0$, that $\dfrac{f(x) - f(x_0)}{x - x_0}$ has a limit equal to $+\infty$ or to $-\infty$; one then says that f is differentiable at x_0 and has derivative $+\infty$ (resp. $-\infty$) there; if the function f has a derivative $f'(x)$ (finite or infinite) at every point x of I, then the function f' (with values in $\overline{\mathbf{R}}$) is again called the derived function (or simply the derivative) of f. One generalizes the definitions of right and left derivative similarly.

Example. At the point $x = 0$ the function $x^{1/3}$ (the inverse function of x^3, a homeomorphism of **R** onto itself) has a derivative, equal to $+\infty$; at $x = 0$ the function $|x|^{1/3}$ has right derivative $+\infty$ and left derivative $-\infty$.

The formulae for the derivative of a sum, of a product of differentiable real functions, and for the inverse of a differentiable function (props. 1, 3 and 4), as well as for the derivative of a (real-valued) composition of functions (prop. 5) remain valid when the derivatives that occur are infinite, so long as all the expressions that occur in these formulae make sense (*Gen. Top.*, IV, p. 345–346). In fact, if in prop. 6 one supposes that f is strictly increasing (resp. strictly decreasing) and continuous on I, and if $f'(x_0) = 0$, then the inverse function g has a derivative equal to $+\infty$

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