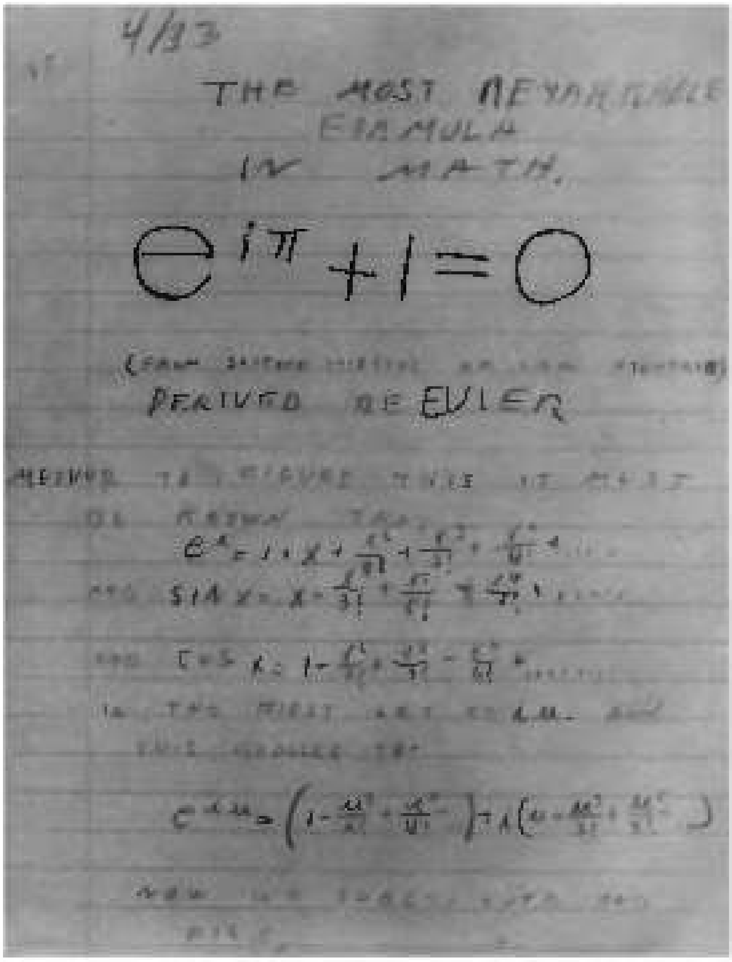
Dr. Euler's Fabulous Formula



In an entry made in one of his teenage notebooks in April 1933, just before his fifteenth birthday, the future physics Nobel prize winner Richard Feynman (1918–1988) took notice of a major theme of this book. Notice the power series expansions for the exponential, sine, and cosine functions immediately below the "most remarkable result in math." The next line is the start of the standard derivation of Euler's formula (also known as Euler's identity) $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$, of which the "remarkable result" is the special case of $\alpha = \pi$. (Feynman's source, *The Science History of the Universe*, was a ten-volume reference set first published in 1909.) Although remembered today as a physicist, Feynman was also a talented mathematician, who wrote in his *The Character of Physical Law* (1965), "To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature. … If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in." Feynman would surely have agreed with one of the early working titles to this book: *Complex Numbers Are Real!* (Photograph courtesy of the Archives, California Institute of Technology.)

Dr. Euler's Fabulous Formula

CURES MANY MATHEMATICAL ILLS

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For Patricia Ann

who (like Euler's formula) is both complex and beautiful

May the God who watches over the right use of
mathematical symbols in manuscript, print,
and on the blackboard, forgive me [my sins].

Hermann Weyl, Professor of Mathematics from 1933
to 1952 at the Institute for Advanced Study, in his book
The Classical Groups, Princeton 1946, p. 289

Contents

Chapter 2. Vector Trips
(some complex plane problems in which direction matters)

Chapter 3. The Irrationality of π^2
("higher" math at the sophomore level)

Chapter 4. Fourier Series
(named after Fourier but Euler was there first—but he was, alas, partially WRONG!)

Chapter 5. Fourier Integrals
(what happens as the period of a periodic function becomes infinite, and other neat stuff)

Chapter 6. Electronics and $\sqrt{-1}$
(technological applications of complex numbers that Euler, who was a practical fellow himself, would have loved)

What This Book Is About, What You Need to Know to Read It, and WHY You Should Read It

Everything of any importance is founded on mathematics.
—Robert Heinlein, *Starship Troopers* (1959)

Several years ago Princeton University Press published my *An Imaginary Tale: The Story of* $\sqrt{-1}$ (1998), which describes the agonizingly long, painful discovery of complex numbers. Historical in spirit, that book still had a lot of mathematics in it. And yet, there was so much I had to leave out or else the book would have been twice its size. This book is much of that "second half" I had to skip by in 1998. While there is some historical discussion here, too, the emphasis is now on more advanced mathematical arguments (but none beyond the skills I mention below), on issues that I think could fairly be called the "sexy part" of complex numbers. There is, of course, some overlap between the two books, but whenever possible I have referred to results derived in *An Imaginary Tale* and have not rederived them here.

To read this book you should have a mathematical background equivalent to what a beginning third year college undergraduate in an engineering or physics program of study would have completed. That is, two years of calculus, a first course in differential equations, and perhaps some preliminary acquitance with matrix algebra and elementary probability. Third year math majors would *certainly* have the required background! These requirements will, admittedly, leave more than a few otherwise educated readers out in the cold. Such people commonly share the attitude of Second World War British Prime Minister Winston Churchill, who wrote the following passage in his 1930 autobiographical work *My Early Years: A Roving Commission*:

> I had a feeling once about Mathematics, that I saw it all—Depth beyond depth was revealed to me—the Byss and the Abyss. I saw, as one might see the transit of Venus—or even the Lord Mayor's Show, a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how it happened and why the tergiversation was inevitable: and how the one step involved all the others. It was like politics. But it was after dinner and I let it go!

Churchill was, I'm sure, mostly trying to be funny, but others, equally frank about their lack of mathematical knowledge, seem not to be terribly concerned about it. As an example, consider a review by novelist Joyce Carol Oates of E. L. Doctorow's 2000 novel *City of God* (*New York Review of Books*, March 9, 2000, p. 31). Oates (a professor at Princeton and a winner of the Pulitzer Prize) wrote, "The sciences of the universe are disciplines whose primary language is mathematics, not conventional speech, and it's inaccessible even to the reasonably educated non-mathematician." I disagree. Shouldn't being ignorant of what is taught each year to a *million college freshmen and sophomores* (math, at the level of this book) world-wide, the vast majority of whom are *not* math majors, be reason for at least a *little* concern?

Some of Oates's *own* literary colleagues would also surely disagree with her. As novelist Rebecca Goldstein wrote in her 1993 work *Strange Attractors*, "Mathematics and music *are* God's languages. When you speak them . . . you're speaking directly to God." I am also thinking of such past great American poets as Henry Longfellow and Edna St. Vincent Millay. It was Millay, of course, who wrote the often quoted line from her 1923 *The Harp-Weaver*, "Euclid alone has looked on Beauty bare." But it was Longfellow who long ago really laid it on the line, who really put his finger on the knowledge gap that exists without embarrassment in many otherwise educated minds. In the opening passages of Chapter 4 in his 1849 novella *Kavanagh, A Tale*, the dreamy and thoughtful schoolmaster Mr. Churchill and his wife Mary have the following exchange in his study:

> "For my part [says Mary] I do not see how you can make mathematics poetical. There is no poetry in them."
>
> "Ah [exclaims Mr. Churchill], that is a very great mistake! There is something divine in the science of numbers. Like God, it holds

> the sea in the hollow of its hand. It measures the earth; it weighs the stars; it illuminates the universe; it is law, it is order, it is beauty. And yet we imagine—that is, most of us—that its highest end and culminating point is book-keeping by double entry. It is our way of teaching it that makes it so prosaic."

You, of course, since you are reading this book, fully appreciate and agree with *this* Mr. Churchill's words!

Preface

"When Did Math Become Sexy?"

The above question, from a 2002 editorial[1] in The *Boston Globe*, observes how the concept of beauty in mathematics has moved from the insulated male-dominated world of pipe-smoking, sherry-sipping mathematicians in tweedy coats and corduroy trousers, at the weekly afternoon college seminar, to the "real world" of truck drivers, teenagers, and retired couples looking for a bit of entertainment on a rainy afternoon. You'll see what I mean if you watch the film *Spider-Man 2* (2004); look for Tobey Maguire's casual reference in a Hollywood super-hero adventure flick to Bernoulli's solution to the famous problem of determining the minimum gravitational descent time curve.

In support of its claim, the *Globe* editorial cites three plays and a movie as examples of this remarkable intellectual transition. In the play *Copenhagen* we see a dramatic presentation of a debate between the physicists Niels Bohr and Werner Heisenberg on quantum mechanics. Heisenberg, who gave his very name to the mathematics of inherent uncertainty in nature (discussed in chapter 5), talks at one point of his first understanding of the new quantum theory: it is, he says "A world of pure mathematical structures. I'm too excited to sleep." He then, as the *Globe* put it, "rushes out in the dawn to climb a rock jutting out to sea, the crashing surf all around." It seems a scene we've all seen many times before in films from the 1930s and 1940s, just before (or after) the heroine is bedded. The erotic connection between mathematical insight and sexual orgasm is simply impossible to deny.[2]

The editorial then goes on to discuss the plays *Proof* (in which arcane formulas are presented as "beautiful"), *Q.E.D.* (about the theoretical physicist Richard Feynman, who often talked of the wondrous way mathematics is at the root of any meaningful interpretation of nature), and Ron Howard's Oscar-winning 2001 film *A Beautiful Mind*. While that film was an interpretation of the life of Princeton mathematician John Nash as viewed through a somewhat distorted glass, it did the best a Hollywood movie probably could do in telling a general audience (from teenagers on up) of what Nash's work in game theory was (sort of) all about. Oddly, the *Globe* didn't mention the 1997 film *Good Will Hunting* (I say *oddly* because the stars are Ben Affleck and Matt Damon, two of Boston's own), in which line after line of Fourier integral equations fills the screen during the opening sequences. That film, also an Oscar-winner, also has a mathematical genius—a handsome night-shift custodian at MIT—as its hero. In a powerful emotional reversal of the idea that math is wonderful, Tom Hanks's mob hitman-on-the-run in *The Road to Perdition* (2002) finds common ground with his son when the two discover that they both *hate* math. As poets are so fond of saying, hate and love are two sides of the same coin and so, even in this violent film, math plays an emotional role as male-bonding glue.

Even before the examples mentioned by the *Globe*, mathematics had played a prominent role in a number of mainstream films.[3] Take a look at such movies as *Straw Dogs* (1971), *It's My Turn* (1980), *Stand and Deliver* (1987), *Sneakers* (1992), *The Mirror Has Two Faces* (1996), *Contact* (1997), *Pi* (1998), and *Enigma* (2002), and you'll agree that the *Globe* was right—mathematics (often equated with extreme dorkiness) *has* become sexy! Even television has gotten into the act, with the 2005 series *Numb3rs*, involving an FBI agent with a mathematical genius brother who helps solve crime mysteries (the show's technical adviser is a professor of mathematics at Caltech, where numerous "atmosphere" scenes were shot to achieve the appropriate academic ambiance).

The *Globe* thought this embracing of mathematics by popular culture had happened because: "The attraction of math and science is that they require embracing the unknowable." The inclusion of science in this statement is interesting, because many physicists think the most beautiful equations (note the plural) are those of Einstein's theory of gravity. For them it is not the mathematics, itself, that is the source of the beauty, but rather the equations' elegant expression of physical reality. For them the mathematics is the visible flesh, yes, but it is the *physics* that is the soul—and source—of the beauty. The 1933 Physics Nobel prize co-winner Paul Dirac was a notable exponent of that view. Dirac (1902–1984) was famous for his many comments on *technical* beauty[4]; in response to being asked (in Moscow, in 1955) about his philosophy of physics, for example, he wrote on a blackboard "Physical Laws should have mathematical beauty." That blackboard is preserved to this day by admiring Russian physicists.

Of course, as physicists learn more physics their equations change. No one, not even Einstein, is immune to this evolution. Just as Newton's gravitational theory gave way to Einstein's, Einstein's is having to give way to newer ideas that are compatible, as Einstein's equations are not, with quantum mechanics. So, Einstein's physics is, at some fundamentally deep level, "wrong" (or, more graciously, "missing something") and so is only approximately correct. But does that mean the mathematical beauty of the equations in Einstein's theory fades?

I don't think so. In the Introduction I'll discuss a number of views that various writers have put forth on what makes theories (and their equations) beautiful, but one point that isn't mentioned there is perhaps best done so here. The reason I think Einstein's theory is still beautiful, even though we now know it cannot possibly be perfectly correct, is that it is the result of *disciplined reasoning*. Einstein created new physics, yes, but not just willy nilly. His work was done while satisfying certain severe restrictions. For example, the physical laws of nature must be the same for all observers, no matter what may be their state of motion in the universe. A theory that satisfies such a broad constraint *must*, I think, be beautiful.

Ugly creations, in my opinion, be they theories or paintings, are ones that obey no constraints, that have no discipline in their nature. It is by that criterion, alone, for example, that I place Norman Rockwell *far* above Jackson Pollock as an artist. This will no doubt send most modern art fans into near-fatal convulsions and brand me a cultural Neanderthal (my art historian wife's opinion), but anybody who can observe the result of simply throwing paint on a canvas[7]—what two-year-olds routinely do in ten thousand day-care centers every day (my gosh, what *I* do every time I paint a ceiling[8])—and call the outcome art, much less beautiful art, is delusional or at least deeply confused (in my humble opinion). To take my point to the limit, I find the imagery of Jackson Pollock fans exclaiming in awe at the mess formed by paint randomly dripping on the *floor* of the Sistine Chapel, rather than at what Michelangelo *painstakingly*, with *skill* and *discipline*, applied to the ceiling, hilarious. Pollock fans might very well rebut me by saying his works *are* beautiful because he *did* have discipline—the "discipline" never to be hamstrung by discipline! I have heard that argument before from college students, and I must admit I am still trying to come up with a good reply other than rolling my eyes.

In this book the gold standard for mathematical beauty is one of the formulas at the heart of complex number analysis, Euler's formula or identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, where $i = \sqrt{-1}$. The special case of $\theta = \pi$ gives $e^{i\pi} = -1$ or, as it is usually written, $e^{i\pi} + 1 = 0$, a compact expression that I think is of exquisite beauty. I think $e^{i\pi} + 1 = 0$ is beautiful because it is true even in the face of *enormous* potential constraint. The equality is precise; the left-hand side is not "almost" or "pretty near" or "just about" zero, but *exactly* zero. That *five* numbers, each with vastly different origins, and each with roles in mathematics that cannot be exaggerated, should be connected by such a simple relationship, is just stunning. *It is beautiful.* And unlike the physics or chemistry or engineering of today, which will almost surely appear archaic to technicians of the far future, Euler's formula will still appear, to the arbitrarily advanced mathematicians ten thousand years hence, to be beautiful and stunning and untarnished by time.

The great German mathematician Hermann Weyl (1885–1955) is famous for declaring, in only a half-joking way, "My work always tried to unite the truth with the beautiful, but when I had to chose one or the other, I usually chose the beautiful." Read on, and I'll try to demonstrate what Weyl meant by showing you some really beautiful ("sexy"?) complex number calculations, many based in part on Euler's formula.

Introduction

> Like a Shakespearean sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler's equation reaches down into the very depths of existence.
>
> Keith Devlin writing of $e^{i\pi}+1=0$[1]

The nineteenth century Harvard mathematician Benjamin Peirce (1809–1880) made a tremendous impression on his students. As one of them wrote many years after Peirce's death, "The appearance of Professor Benjamin Peirce, whose long gray hair, straggling grizzled beard and unusually bright eyes sparkling under a soft felt hat, as he walked briskly but rather ungracefully across the college yard, fitted very well with the opinion current among us that we were looking upon a real live genius, who had a touch of the prophet in his make-up."[2] That same former student went on to recall that during one lecture "he established the relation connecting π, e, and i, $e^{\pi/2}=\sqrt[i]{i}$, which evidently had a strong hold on his imagination.[3] He dropped his chalk and rubber (i.e., eraser), put his hands in his pockets, and after contemplating the formula a few minutes turned to his class and said very slowly and impressively, 'Gentlemen, that is surely true, it is absolutely paradoxical, we can't understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth.'"

Like any good teacher, Peirce was almost certainly striving to be dramatic ("Although we could rarely follow him, we certainly sat up and took notice"), but with those particular words he reached too far. We certainly *can* understand what Peirce always called the "mysterious formula," and we certainly *do* know what it means. But, yes, it *is* still a wonderful, indeed beautiful, expression; no amount of "understanding" can ever diminish its power to awe us. As one limerick (a literary form particularly beloved by mathematicians) puts it,

> *e* raised to the *pi* times *i*,
> And plus 1 leaves you nought but a sigh.
> This fact amazed Euler
> That genius toiler,
> And still gives us pause, bye the bye.

The limerick puts front-and-center several items we need to discuss pretty soon. What are *e*, *pi*, and *i*, and who was Euler? Now, it is hard for me to believe that there are any literate readers in the world who haven't heard of the transcendental numbers $e = 2.71828182\ldots$ and $pi = \pi = 3.14159265\ldots$, and of the imaginary number $i = \sqrt{-1}$. As for Euler, he was surely one of the greatest of all mathematicians. Making lists of the "greatest" is a popular activity these days, and I would wager that the Swiss-born Leonhard Euler (1707–1783) would appear somewhere among the top five mathematicians of all time on the list made by any mathematician in the world today (Archimedes, Newton, and Gauss would give him stiff competition, but what great company *they* are!).

Now, before I launch into the particulars of e, π, and $\sqrt{-1}$, what about the stupefying audacity I displayed in the Preface by declaring $e^{i\pi} + 1 = 0$ to be "an expression of exquisite beauty"? I didn't do that lightly and, indeed, I have "official authority." In the fall 1988 issue of the *Mathematical Intelligencer*, a scholarly quarterly journal of mathematics sponsored by the prestigious publisher of mathematics books and journals, Springer-Verlag, there was the call for a vote on the most beautiful theorem in mathematics. Readers of the *Intelligencer*, consisting almost entirely of academic and industrial mathematicians, were asked to rank twenty-four given theorems on a scale of 0 to 10, with 10 being the most beautiful and 0 the least. The list contained, in addition to $e^{i\pi} + 1 = 0$, such seminal theorems as

(a) The number of primes is infinite;
(b) There is no rational number whose square is 2;

(c) π is transcendental;
(d) A continuous mapping of the closed unit disk into itself has a fixed point.

A distinguished list, indeed.

The results, from a total of 68 responses, were announced in the summer 1990 issue. Receiving the top average score of 7.7 was $e^{i\pi}+1=0$. The scores for the other theorems above, by comparison, were 7.5 for (a), 6.7 for (b), 6.5 for (c), and 6.8 for (d). The lowest ranked theorem (a result in number theory by the Indian genius Ramanujan) received an average score of 3.9. So, it is *official*: $e^{i\pi}+1=0$ *is* the most beautiful equation in mathematics! (I hope most readers can see my tongue stuck firmly in my cheek as they read these words, and will not send me outraged e-mails to tell me why their favorite expression is so much *more* beautiful.)

Of course, the language used above is pretty sloppy, because $e^{i\pi}+1=0$ is actually *not* an equation. An equation (in a single variable) is a mathematical expression of the form $f(x)=0$, for example, $x^2+x-2=0$, which is true only for certain values of the variable, that is, for the *solutions* of the equation. For the just cited quadratic equation, for example, $f(x)$ equals zero for the two values of $x=-2$ and $x=1$, *only*. There is no x, however, to solve for in $e^{i\pi}+1=0$. So, it isn't an equation. It isn't an identity, either, like Euler's identity $e^{i\theta}=\cos(\theta)+i\sin(\theta)$, where θ is *any* angle, not just π radians. That's what an *identity* (in a single variable) is, of course, a statement that is *identically* [illegible]

[illegible] There is no variable at all anywhere in $e^{i\pi}+1=0$; just five constants. (Euler's identity is at the heart of this book and it will be established in Chapter 1.) So, $e^{i\pi}+1=0$ is not an equation and it isn't an identity. Well, then, what *is* it? It is a *formula*, a *theorem*.

More to the point for us, here, isn't semantics but rather the issue first raised in the Preface, that of *beauty*. What could it possibly mean to say a mathematical statement is "beautiful"? To that I reply, what does it mean to say a kitten asleep, or an eagle in flight, or a horse in full gallop, or a laughing baby, or . . . is beautiful? An easy answer is that it is all simply in the eye of the beholder (the ultimate "explanation," I suppose, for the popularity of Jackson Pollock's drip paintings), but I think (a in the mathematical case) that there are deeper possibilities. The author of the *Intelligencer* poll (David Wells, the writer of a number of popular mathematical works), for example, offered several good suggestions as to what makes a mathematical expression *beautiful*.

To be beautiful, Wells writes, a mathematical statement must be simple, brief, important, and, obvious when it is stated but perhaps easy to overlook otherwise, *surprising*. (A similar list was given earlier by H. E. Huntley in his 1970 book *The Divine Proportion*.) I think Euler's identity (and its offspring $e^{i\pi} + 1 = 0$) scores high on all four counts, and I believe you will think so too by the end of this book. Not everyone agrees, however, which should be no surprise—there is *always* someone who doesn't agree with *any* statement! For example, in his interesting essay "Beauty in Mathematics," the French mathematician François Le Lionnais (1901–1984) starts off with high praise, writing of $e^{i\pi} + 1 = 0$ that it

> [E]stablishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics, 1, π, and e [for some reason 0 and i are ignored by Le Lionnais]. It was generally considered "the most important formula of mathematics."[1]

But then comes the tomato surprise, with a very big splat in the face: "Today the intrinsic reason for this compatibility has become so obvious[!] that the same formula seems, if not insipid, at least entirely natural."

Well, good for François and his fabulous powers of insight (or is it hindsight?), but such a statement is rightfully greeted with the same skepticism that most mathematicians give to claims from those who say they can "see geometrical shapes in the fourth dimension." Such people only *think* they do. They are certainly "seeing things," all right, but I doubt very much it's the true geometry of hyperspace. When you are finished here, $e^{i\pi} + 1 = 0$ *will* be "obvious," but borderline *insipid*? Never!

At this point, for completeness, I should mention that the great English mathematician G. H. Hardy (1887–1947) had a very odd view of what constitutes beauty in mathematics: to be beautiful, mathematics