

Westminster  
July 2, 1850

equations of equilibrium

My dear Stokes  
As I have not  
any other  
reference for  
purpose, by me,  
be much oblig

on my way from  
Cambridge that made  
me take up the subject

**Michael Spivak**

# CALCULUS

# ON

# MANIFOLDS

Do you know that the  
condition that a differential  
 $\gamma dx$  may be the diff<sup>l</sup> of  
for all points of a surface  
is  $l(\frac{dx}{dz} - \frac{dy}{dz}) + m(\frac{dy}{dz} - \frac{dz}{dz}) + n(\frac{dz}{dz} - \frac{dx}{dz})$   
= 0. I made this out some  
weeks ago with ref<sup>ce</sup> to  
magnetics. With ref<sup>ce</sup> to  
elastic sol<sup>ids</sup> the cond<sup>n</sup> may  
be written - the resultant ar<sup>y</sup>  
at any point of the  
surface must be normal.

Yours very truly  
William Thomson

parallel  
in a very in-  
ray. It was reading  
on diffraction

P.S. The following is also interesting,  
of importance with reference to  
both physical subjects.

$$\int (\alpha dx + \beta dy + \gamma dz) = \pm \int \{ l(\frac{dx}{dz} - \frac{dy}{dz}) + m(\frac{dy}{dz} - \frac{dz}{dz}) + n(\frac{dz}{dz} - \frac{dx}{dz}) \} dS$$

where  $l, m, n$  denote the dir<sup>ct</sup> cosines of normal  
through any el<sup>l</sup>  $dS$  of a surface, & the integ<sup>n</sup>  
in the sec<sup>d</sup> member is performed over a portion



Westminster  
July 2, 1850

equations of equilibrium  
in crystalline bodies

My dear Stokes

on my way from Cambridge that made me take up the subject

As I have not my former paper in *Phil. Mag.* & *Notes on Elastic Solids*

now that the condition that a displacement  $\delta x$  may be the diff. of a

**Michael Spivak**

# CALCULUS

my other reference for purpose, by me

for all points of a surface is  $l(\frac{dx}{dz} - \frac{dy}{dz}) + m(\frac{dy}{dz} - \frac{dx}{dz}) + n(\frac{dx}{dz} - \frac{dy}{dz}) = 0$

# ON

be much obliged

I made this out some weeks ago with ref. to elastic solids the cond. may be written - the resultant arising at any point of the surface must be normal!

# MANIFOLDS

parallel in a very irregular way. It was reading on diffraction

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$$\int (\alpha dx + \beta dy + \gamma dz) = \pm \int \{l(\frac{dx}{dz} - \frac{dy}{dz}) + m(\frac{dy}{dz} - \frac{dx}{dz}) + n(\frac{dx}{dz} - \frac{dy}{dz})\} dS$$

where  $l, m, n$  denote the dir. cosines of normal through any el.  $dS$  of a surface, & the integ. in the sec. member is performed over a portion





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***Michael Spivak***

*Brandeis University*

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# *Calculus on Manifolds*

A MODERN APPROACH TO CLASSICAL THEOREMS  
OF ADVANCED CALCULUS

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The Advanced Book Program



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**MATHEMATICS MONOGRAPH SERIES**

EDITORS: **Robert Gunning**, *Princeton University*;  
**Hugo Rossi**, *Brandeis University*

**Frederick J. Almgren, Jr.**, *Institute for Advanced Study*

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AN INTRODUCTION TO FOURIER SERIES

**Michael Spivak**, *Brandeis University*

CALCULUS ON MANIFOLDS: A MODERN APPROACH TO CLASSICAL

THEOREMS OF ADVANCED CALCULUS

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## *Editors' Foreword*

Mathematics has been expanding in all directions at a fabulous rate during the past half century. New fields have emerged, the diffusion into other disciplines has proceeded apace, and our knowledge of the classical areas has grown ever more profound. At the same time, one of the most striking trends in modern mathematics is the constantly increasing interrelationship between its various branches. Thus the present-day students of mathematics are faced with an immense mountain of material. In addition to the traditional areas of mathematics as presented in the traditional manner—and these presentations do abound—there are the new and often enlightening ways of looking at these traditional areas, and also the vast new areas teeming with potentialities. Much of this new material is scattered indigestibly throughout the research journals, and frequently coherently organized only in the minds and unpublished notes of the working mathematicians. And students desperately need to learn more and more of this material.

This series of brief topical booklets has been conceived as a possible means to tackle and hopefully to alleviate some of these pedagogical problems. They are being written by active research mathematicians, who can look at the latest developments, who can use these developments to clarify and condense the required material, who know what ideas to underscore and what techniques to stress. We hope that they will also serve to present to the able undergraduate an introduction to contemporary research and problems in mathematics, and that they will be sufficiently informal that the personal tastes and attitudes of the leaders in modern mathematics will shine through clearly to the readers.

The area of differential geometry is one in which recent developments have effected great changes. That part of differential geometry centered about Stokes' Theorem, sometimes called the fundamental theorem of multivariate calculus, is traditionally taught in advanced calculus courses (second or third year) and is essential in engineering and physics as well as in several current and important branches of mathematics. However, the teaching of this material has been relatively little affected by the modern developments; so the mathematicians must relearn the material in graduate school, and other scientists are frequently altogether deprived of it. Dr. Spivak's book should be a help to those who wish to see Stokes' Theorem as the modern working mathematician sees it. A student with a good course in calculus and linear algebra behind him should find this book quite accessible.

*Robert Gunnin*

*Hugo Ros*

*Princeton, New Jersey*  
*Waltham, Massachusetts*  
*August 1965*

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## *Preface*

This little book is especially concerned with those portions of “advanced calculus” in which the subtlety of the concepts and methods makes rigor difficult to attain at an elementary level. The approach taken here uses elementary versions of modern methods found in sophisticated mathematics. The formal prerequisites include only a term of linear algebra, a nodding acquaintance with the notation of set theory, and a respectable first-year calculus course (one which at least mentions the least upper bound (sup) and greatest lower bound (inf) of a set of real numbers). Beyond this a certain (perhaps latent) rapport with abstract mathematics will be found almost essential.

The first half of the book covers that simple part of advanced calculus which generalizes elementary calculus to higher dimensions. Chapter 1 contains preliminaries, and Chapters 2 and 3 treat differentiation and integration.

The remainder of the book is devoted to the study of curves, surfaces, and higher-dimensional analogues. Here the modern and classical treatments pursue quite different routes; there are, of course, many points of contact, and a significant encounter occurs in the last section. The very classic equation reproduced on the cover appears also as the last theorem of the book. This theorem (Stokes' Theorem) has had a curious history and has undergone a striking metamorphosis.

The first statement of the Theorem appears as a postscript to a letter, dated July 2, 1850, from Sir William Thomson (Lord Kelvin) to Stokes. It appeared publicly as question 8 on the Smith's Prize Examination for 1854. This competitive examination, which was taken annually by the best mathematics students at Cambridge University, was set from 1849 to 1882 by Professor Stokes; by the time of his death the result was known universally as Stokes' Theorem. At least three proofs were given by his contemporaries: Thomson published one, another appeared in Thomson and Tait's *Treatise on Natural Philosophy*, and Maxwell provided another in *Electricity and Magnetism* [13]. Since this time the name of Stokes has been applied to much more general results, which have figured so prominently in the development of certain parts of mathematics that Stokes' Theorem may be considered a case study in the value of generalization.

In this book there are three forms of Stokes' Theorem. The version known to Stokes appears in the last section, along with its inseparable companions, Green's Theorem and the Divergence Theorem. These three theorems, the classical theorems of the subtitle, are derived quite easily from a modern Stokes' Theorem which appears earlier in Chapter 5. What the classical theorems state for curves and surfaces, this theorem states for the higher-dimensional analogues (manifolds) which are studied thoroughly in the first part of Chapter

5. This study of manifolds, which could be justified solely on the basis of their importance in modern mathematics, actually involves no more effort than a careful study of curves and surfaces alone would require.

The reader probably suspects that the modern Stokes' Theorem is at least as difficult as the classical theorems derived from it. On the contrary, it is a very simple consequence of yet another version of Stokes' Theorem; this very abstract version is the final and main result of Chapter 4. It is entirely reasonable to suppose that the difficulties so far avoided must be hidden here. Yet the proof of this theorem is, in the mathematician's sense, an utter triviality—a straight-forward computation. On the other hand, even the statement of this triviality cannot be understood without a horde of difficu-

definitions from Chapter 4. There are good reasons why the theorems should all be easy and the definitions hard. As the evolution of Stokes' Theorem revealed, a single simple principle, can masquerade as several difficult results; the proofs of many theorems involve merely stripping away the disguise. The definitions, on the other hand, serve a twofold purpose: they are rigorous replacements for vague notions, and machinery for elegant proofs. The first two sections of Chapter 4 define precisely, and prove the rules for manipulating, what are classically described as "expressions of the form"  $P dx + Q dy + R dz$ , or  $P dx dy + Q dy dz + R dz dx$ . Chains, defined in the third section, and partitions of unity (already introduced in Chapter 3) free our proofs from the necessity of chopping manifolds up into small pieces; they reduce questions about manifolds, where everything seems hard, to questions about Euclidean space, where everything is easy.

Concentrating the depth of a subject in the definitions is undeniably economical, but it is bound to produce some difficulties for the student. I hope the reader will be encouraged to learn Chapter 4 thoroughly by the assurance that the results will justify the effort: the classical theorems of the last section represent only a few, and by no means the most important, applications of Chapter 4; many others appear as problems, and further developments will be found by exploring the bibliography.

The problems and the bibliography both deserve a few words. Problems appear after every section and are numbered (like the theorems) within chapters. I have starred those problems whose results are used in the text, but this precaution should be unnecessary—the problems are the most important parts of the book, and the reader should at least attempt them all. It was necessary to make the bibliography either very incomplete or unwieldy, since half the major branches of mathematics could legitimately be recommended as reasonable continuations of the material in the book. I have tried to make it incomplete but tempting.

Many criticisms and suggestions were offered during the writing of this book. I am particularly grateful to Richard Palais, Hugo Rossi, Robert Seeley, and Charles Stenard for their many helpful comments.

I have used this printing as an opportunity to correct many misprints and minor errors pointed out to me by indulgent readers. In addition, the material following Theorem 3-11 has been completely revised and corrected. Other important changes, which could not be incorporated in the text without excessive alteration, are listed in the Addenda at the end of the book.

***Michael Spivak***

*Waltham, Massachusetts*  
*March 1968*

## *Functions on Euclidean Space*

### **NORM AND INNER PRODUCT**

**Euclidean  $n$ -space  $\mathbf{R}^n$**  is defined as the set of all  $n$ -tuples  $(x^1, \dots, x^n)$  of real numbers  $x^i$  (a “1-tuple numbers” is just a number and  $\mathbf{R}^1 = \mathbf{R}$ , the set of all real numbers). An element of  $\mathbf{R}^n$  is often called a point in  $\mathbf{R}^n$ , and  $\mathbf{R}^1$ ,  $\mathbf{R}^2$ ,  $\mathbf{R}^3$  are often called the line, the plane, and space, respectively. If  $x$  denotes an element of  $\mathbf{R}^n$ , then  $x$  is an  $n$ -tuple of numbers, the  $i$ th one of which is denoted  $x^i$ ; thus we can write

$$x = (x^1, \dots, x^n).$$

A point in  $\mathbf{R}^n$  is frequently also called a vector in  $\mathbf{R}^n$ , because  $\mathbf{R}^n$ , with  $x + y = (x^1 + y^1, \dots, x^n + y^n)$  and  $ax = (ax^1, \dots, ax^n)$ , as operations, is a vector space (over the real numbers, of dimension  $n$ ). In this vector space there is the notion of the length of a vector  $x$ , usually called the **norm**  $|x|$  of  $x$  and defined by  $|x| = \sqrt{(x^1)^2 + \dots + (x^n)^2}$ . If  $n = 1$ , then  $|x|$  is the usual absolute value of  $x$ . The relationship between the norm and the vector space structure of  $\mathbf{R}^n$  is very important.

**1-1 Theorem.** *If  $x, y \in \mathbf{R}^n$  and  $a \in \mathbf{R}$ , then*

1.  $|x| \geq 0$ , and  $|x| = 0$  if and only if  $x = 0$ .
2.  $|\sum_{i=1}^n x^i y^i| \leq |x| \cdot |y|$ ; equality holds if and only if  $x$  and  $y$  are linearly dependent.
3.  $|x + y| \leq |x| + |y|$ .
4.  $|ax| = |a| \cdot |x|$ .

**Proof**

1. is left to the reader.
2. If  $x$  and  $y$  are linearly dependent, equality clearly holds. If not, then  $\lambda y - x \neq 0$  for all  $\lambda \in \mathbf{R}$ , so

$$\begin{aligned} 0 < |\lambda y - x|^2 &= \sum_{i=1}^n (\lambda y^i - x^i)^2 \\ &= \lambda^2 \sum_{i=1}^n (y^i)^2 - 2\lambda \sum_{i=1}^n x^i y^i + \sum_{i=1}^n (x^i)^2. \end{aligned}$$

Therefore the right side is a quadratic equation in  $\lambda$  with no real solution, and its discriminant must be negative. Thus

$$4 \left( \sum_{i=1}^n x^i y^i \right)^2 - 4 \sum_{i=1}^n (x^i)^2 \cdot \sum_{i=1}^n (y^i)^2 < 0.$$

$$\begin{aligned}
 |x + y|^2 &= \sum_{i=1}^n (x^i + y^i)^2 \\
 &= \sum_{i=1}^n (x^i)^2 + \sum_{i=1}^n (y^i)^2 + 2\sum_{i=1}^n x^i y^i \\
 &\leq |x|^2 + |y|^2 + 2|x| \cdot |y| \quad \text{by (2)} \\
 &= (|x| + |y|)^2.
 \end{aligned}$$

3.

$$4. |ax| = \sqrt{\sum_{i=1}^n (ax^i)^2} = \sqrt{a^2 \sum_{i=1}^n (x^i)^2} = |a| \cdot |x|.$$

The quantity  $\sum_{i=1}^n x^i y^i$  which appears in (2) is called the **inner product** of  $x$  and  $y$  and denoted  $\langle x, y \rangle$ . The most important properties of the inner product are the following.

**1-2 Theorem.** If  $x, x_1, x_2$  and  $y, y_1, y_2$  are vectors in  $\mathbf{R}^n$  and  $a \in \mathbf{R}$ , then

1.  $\langle x, y \rangle = \langle y, x \rangle$

(symmetry)

2.  $\langle ax, y \rangle = \langle x, ay \rangle = a\langle x, y \rangle$

(bilinearity)

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle \quad \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$$

3.  $\langle x, x \rangle \geq 0$ , and  $\langle x, x \rangle = 0$  if and

(positive definiteness)

only if  $x = 0$

4.  $|x| = \sqrt{\langle x, x \rangle}$ .

5.  $\langle x, y \rangle = \frac{|x + y|^2 - |x - y|^2}{4}$

(polarization identity)

### Proof

1.  $\langle x, y \rangle = \sum_{i=1}^n x^i y^i = \sum_{i=1}^n y^i x^i = \langle y, x \rangle$ .

2. By (1) it suffices to prove

$$\begin{aligned}
 \langle ax, y \rangle &= a\langle x, y \rangle, \\
 \langle x_1 + x_2, y \rangle &= \langle x_1, y \rangle + \langle x_2, y \rangle.
 \end{aligned}$$

These follow from the equations

$$\begin{aligned}
 \langle ax, y \rangle &= \sum_{i=1}^n (ax^i)y^i = a \sum_{i=1}^n x^i y^i = a\langle x, y \rangle, \\
 \langle x_1 + x_2, y \rangle &= \sum_{i=1}^n (x_1^i + x_2^i)y^i = \sum_{i=1}^n x_1^i y^i + \sum_{i=1}^n x_2^i y^i \\
 &= \langle x_1, y \rangle + \langle x_2, y \rangle.
 \end{aligned}$$

3. and

4. are left to the reader.

$$\begin{aligned}
 &= \frac{1}{4}[\langle x+y, x+y \rangle - \langle x-y, x-y \rangle] \quad \text{by (4)} \\
 &= \frac{1}{4}[\langle x,x \rangle + 2\langle x,y \rangle + \langle y,y \rangle - (\langle x,x \rangle - 2\langle x,y \rangle + \langle y,y \rangle)] \\
 &= \langle x,y \rangle. \quad \blacksquare
 \end{aligned}$$

We conclude this section with some important remarks about notation. The vector  $(0, \dots, 0)$  will usually be denoted simply  $0$ . The **usual basis** of  $\mathbf{R}^n$  is  $e_1, \dots, e_n$ , where  $e_i = (0, \dots, 1, \dots, 0)$ , with the 1 in the  $i$ th place. If  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is a linear transformation, the matrix of  $T$  with respect to the usual bases of  $\mathbf{R}^n$  and  $\mathbf{R}^m$  is the  $m \times n$  matrix  $A = (a_{ij})$ , where  $T(e_i) = \sum_{j=1}^m a_{ji}e_j$ —the coefficients of  $T(e_i)$  appear in the  $i$ th column of the matrix. If  $S: \mathbf{R}^m \rightarrow \mathbf{R}^p$  has the  $p \times m$  matrix  $B$ , then  $S \circ T$  has the  $p \times n$  matrix  $BA$  [here  $S \circ T(x) = S(T(x))$ ; most books on linear algebra denote  $S \circ T$  simply  $ST$ ]. To find  $T(x)$  one computes the  $m \times 1$  matrix

$$\begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix};$$

then  $T(x) = (y^1, \dots, y^m)$ . One notational convention greatly simplifies many formulas: if  $x \in \mathbf{R}^n$  and  $y \in \mathbf{R}^m$ , then  $(x,y)$  denotes

$$(x^1, \dots, x^n, y^1, \dots, y^m) \in \mathbf{R}^{n+m}.$$

**Problems. 1-1.\*** Prove that  $|x| \leq \sum_{i=1}^n |x^i|$

**1-2.** When does equality hold in Theorem 1-1(3)? *Hint:* Re-examine the proof; the answer is not “when  $x$  and  $y$  are linearly dependent.”

**1-3.** Prove that  $|x - y| \leq |x| + |y|$ . When does equality hold?

**1-4.** Prove that  $||x| - |y|| \leq |x - y|$ .

**1-5.** The quantity  $|y - x|$  is called the **distance** between  $x$  and  $y$ . Prove and interpret geometrically the “triangle inequality”:  $|z - x| \leq |z - y| + |y - x|$ .

**1-6.** Let  $f$  and  $g$  be integrable on  $[a,b]$ .

(a) Prove that  $|\int_a^b f \cdot g| \leq (\int_a^b f^2)^{1/2} \cdot (\int_a^b g^2)^{1/2}$ . *Hint:* Consider separately the cases  $0 = \int_a^b (f - \lambda g)^2$  for some  $\lambda \in \mathbf{R}$  and  $0 < \int_a^b (f - \lambda g)^2$  for all  $\lambda \in \mathbf{R}$

(b) If equality holds, must  $f = \lambda g$  for some  $\lambda \in \mathbf{R}$ ? What if  $f$  and  $g$  are continuous?

(c) Show that Theorem 1-1(2) is a special case of (a).

**1-7.** A linear transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is **norm preserving** if  $|T(x)| = |x|$ , and **inner product preserving** if  $\langle Tx, Ty \rangle = \langle x, y \rangle$ .

(a) Prove that  $T$  is norm preserving if and only if  $T$  is innerproduct preserving.

(b) Prove that such a linear transformation  $T$  is 1-1 and  $T^{-1}$  is of the same sort.

**1-8.** If  $x, y \in \mathbf{R}^n$  are non-zero, the **angle** between  $x$  and  $y$ , denoted  $\angle(x,y)$ , is defined as  $\arccos(\langle x,y \rangle / |x| \cdot |y|)$ , which makes sense by Theorem 1-1 (2). The linear transformation  $T$  is **angle**



**preserving** if  $T$  is 1-1, and for  $x, y \neq 0$  we have  $\angle(Tx, Ty) = \angle(x, y)$ .

(a) Prove that if  $T$  is norm preserving, then  $T$  is angle preserving.

(b) If there is a basis  $x_1, \dots, x_n$  of  $\mathbf{R}^n$  and numbers  $\lambda_1, \dots, \lambda_n$  such that  $Tx_i = \lambda_i x_i$ , prove that  $T$  is angle preserving if and only if all  $|\lambda_i|$  are equal.

(c) What are all angle preserving  $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ ?

**1-9.** If  $0 \leq \theta < \pi$ , let  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  have the matrix  $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ . Show that  $T$  is angle preserving and if  $x \neq 0$ , then  $\angle(x, Tx) = \theta$ .

**1-10.\*** If  $T: \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a linear transformation, show that there is a number  $M$  such that  $|T(h)| \leq M|h|$  for  $h \in \mathbf{R}^m$ . *Hint:* Estimate  $|T(h)|$  in terms of  $|h|$  and the entries in the matrix of  $T$ .

**1-11.** If  $x, y \in \mathbf{R}^n$  and  $z, w \in \mathbf{R}^m$ , show that  $\langle (x, z), (y, w) \rangle = \langle x, y \rangle + \langle z, w \rangle$  and  $|(x, z)| = \sqrt{|x|^2 + |z|^2}$ . Note that  $(x, z)$  and  $(y, w)$  denote points in  $\mathbf{R}^{n+m}$ .

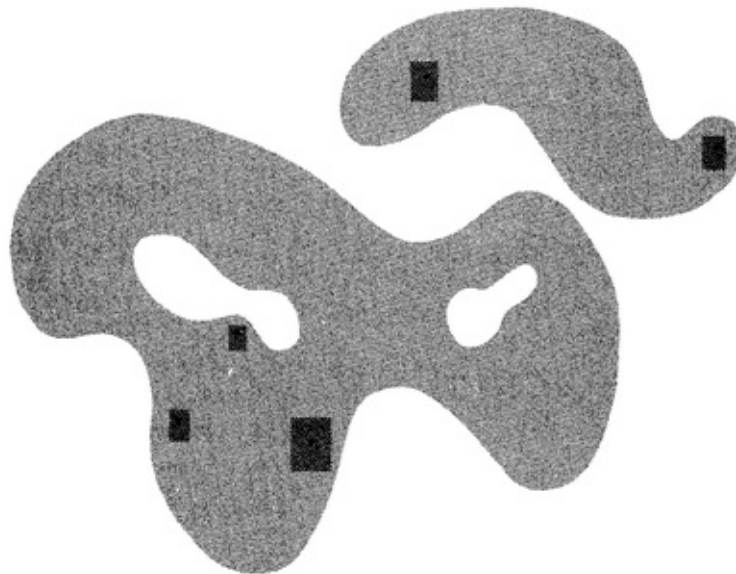
**1-12.\*** Let  $(\mathbf{R}^n)^*$  denote the dual space of the vector space  $\mathbf{R}^n$ . If  $x \in \mathbf{R}^n$ , define  $\phi_x \in (\mathbf{R}^n)^*$  by  $\phi_x(y) = \langle x, y \rangle$ . Define  $T: \mathbf{R}^n \rightarrow (\mathbf{R}^n)^*$  by  $T(x) = \phi_x$ . Show that  $T$  is a 1-1 linear transformation and conclude that every  $\phi \in (\mathbf{R}^n)^*$  is  $\phi_x$  for a unique  $x \in \mathbf{R}^n$ .

**1-13.\*** If  $x, y \in \mathbf{R}^n$ , then  $x$  and  $y$  are called **perpendicular** (or **orthogonal**) if  $\langle x, y \rangle = 0$ . If  $x$  and  $y$  are perpendicular, prove that  $|x + y|^2 = |x|^2 + |y|^2$ .

## ***SUBSETS OF EUCLIDEAN SPACE***

The closed interval  $[a, b]$  has a natural analogue in  $\mathbf{R}^2$ . This is the **closed rectangle**  $[a, b] \times [c, d]$  defined as the collection of all pairs  $(x, y)$  with  $x \in [a, b]$  and  $y \in [c, d]$ . More generally, if  $A \subset \mathbf{R}^m$  and  $B \subset \mathbf{R}^n$ , then  $A \times B \subset \mathbf{R}^{m+n}$  is defined as the set of all  $(x, y) \in \mathbf{R}^{m+n}$  with  $x \in A$  and  $y \in B$ . In particular,  $\mathbf{R}^{m+n} = \mathbf{R}^m \times \mathbf{R}^n$ . If  $A \subset \mathbf{R}^m$ ,  $B \subset \mathbf{R}^n$ , and  $C \subset \mathbf{R}^p$ , then  $(A \times B) \times C = A \times (B \times C)$ , and both of these are denoted simply  $A \times B \times C$ ; this convention is extended to the product of any number of sets. The set  $[a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbf{R}^n$  is called a **closed rectangle** in  $\mathbf{R}^n$ , while the set  $(a_1, b_1) \times \dots \times (a_n, b_n) \subset \mathbf{R}^n$  is called an **open rectangle**. More generally a set  $U \subset \mathbf{R}^n$  is called **open** (Figure 1-1) if for each  $x \in U$  there is an open rectangle  $A$  such that  $x \in A \subset U$ .

A subset  $C$  of  $\mathbf{R}^n$  is **closed** if  $\mathbf{R}^n - C$  is open. For example, if  $C$  contains only finitely many points then  $C$  is closed.

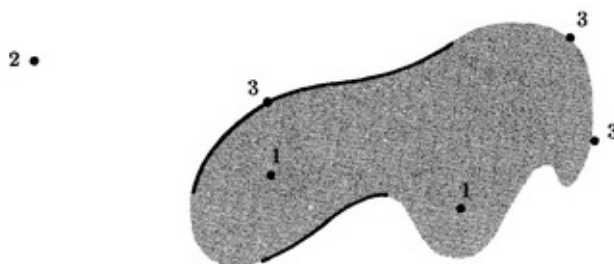


**FIGURE 1-1**

The reader should supply the proof that a closed rectangle in  $\mathbf{R}^n$  is indeed a closed set.

If  $A \subset \mathbf{R}^n$  and  $x \in \mathbf{R}^n$ , then one of three possibilities must hold (Figure 1-2):

1. There is an open rectangle  $B$  such that  $x \in B \subset A$ .
2. There is an open rectangle  $B$  such that  $x \in B \subset \mathbf{R}^n - A$ .
3. If  $B$  is any open rectangle with  $x \in B$ , then  $B$  contains points of both  $A$  and  $\mathbf{R}^n - A$ .



**FIGURE 1-2**

Those points satisfying (1) constitute the **interior** of  $A$ , those satisfying (2) the **exterior** of  $A$ , and those satisfying (3) the **boundary** of  $A$ . Problems 1-16 to 1-18 show that these terms may sometimes have unexpected meanings.

It is not hard to see that the interior of any set  $A$  is open, and the same is true for the exterior of  $A$ , which is, in fact, the interior of  $\mathbf{R}^n - A$ . Thus (Problem 1-14) their union is open, and what remains, the boundary, must be closed.

A collection  $O$  of open sets is an **open cover** of  $A$  (or, briefly, **covers**  $A$ ) if every point  $x \in A$  is in some open set in the collection  $O$ . For example, if  $O$  is the collection of all open intervals  $(a, a + 1)$  for  $a \in \mathbf{R}$ , then  $O$  is a cover of  $\mathbf{R}$ . Clearly no finite number of the open sets in  $O$  will cover  $\mathbf{R}$  or, for that matter, any unbounded subset of  $\mathbf{R}$ . A similar situation can also occur for bounded sets. If  $O$  is the collection of all open intervals  $(1/n, 1 - 1/n)$  for all integers  $n > 1$ , then  $O$  is an open cover of  $(0,1)$ , but again no finite collection of sets in  $O$  will cover  $(0,1)$ . Although this phenomenon may not appear

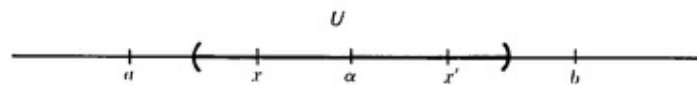
particularly scandalous, sets for which this state of affairs cannot occur are of such importance that they have received a special designation: a set  $A$  is called **compact** if every open cover  $\mathcal{O}$  contains a finite subcollection of open sets which also covers  $A$ .

A set with only finitely many points is obviously compact and so is the infinite set  $A$  which contains 0 and the numbers  $1/n$  for all integers  $n$  (reason: if  $\mathcal{O}$  is a cover, then  $0 \in U$  for some open set  $U$  in  $\mathcal{O}$ ; there are only finitely many other points of  $A$  not in  $U$ , each requiring at most one more open set).

Recognizing compact sets is greatly simplified by the following results, of which only the first has any depth (i.e., uses any facts about the real numbers).

**1-3 Theorem (Heine-Borel).** *The closed interval  $[a,b]$  is compact.*

**Proof.** If  $\mathcal{O}$  is an open cover of  $[a,b]$ , let  $A = \{x: a \leq x \leq b \text{ and } [a,x] \text{ is covered by some finite number of open sets in } \mathcal{O}\}$ .



**FIGURE 1-3**

Note that  $a \in A$  and that  $A$  is clearly bounded above (by  $b$ ). We would like to show that  $b \in A$ . This is done by proving two things about  $\alpha =$  least upper bound of  $A$ ; namely, (1)  $\alpha \in A$  and (2)  $b = \alpha$ .

Since  $\mathcal{O}$  is a cover,  $\alpha \in U$  for some  $U$  in  $\mathcal{O}$ . Then all points in some interval to the left of  $\alpha$  are also in  $U$  (see Figure 1-3). Since  $\alpha$  is the least upper bound of  $A$ , there is an  $x$  in this interval such that  $x \in A$ . Thus  $[a,x]$  is covered by some finite number of open sets of  $\mathcal{O}$ , while  $[x,\alpha]$  is covered by the single set  $U$ . Hence  $[a,\alpha]$  is covered by a finite number of open sets of  $\mathcal{O}$ , and  $\alpha \in A$ . This proves (1).

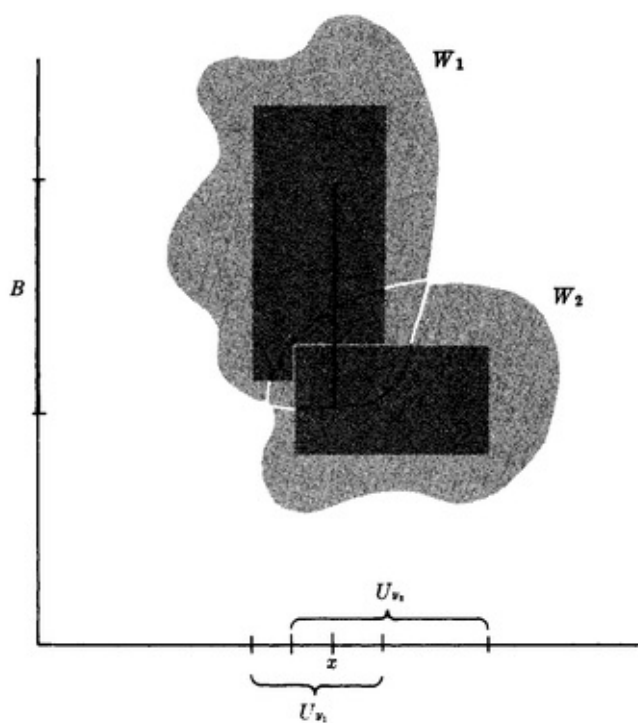
To prove that (2) is true, suppose instead that  $\alpha < b$ . Then there is a point  $x'$  between  $\alpha$  and  $b$  such that  $[\alpha,x'] \subset U$ . Since  $\alpha \in A$ , the interval  $[a,\alpha]$  is covered by finitely many open sets of  $\mathcal{O}$ , while  $[\alpha,x']$  is covered by  $U$ . Hence  $x' \in A$ , contradicting the fact that  $\alpha$  is an upper bound of  $A$ .

If  $B \subset \mathbf{R}^m$  is compact and  $x \in \mathbf{R}^n$ , it is easy to see that  $\{x\} \times B \subset \mathbf{R}^{n+m}$  is compact. However, a much stronger assertion can be made.

**1-4 Theorem.** *If  $B$  is compact and  $\mathcal{O}$  is an open cover of  $\{x\} \times B$ , then there is an open set  $U \subset \mathbf{R}^n$  containing  $x$  such that  $U \times B$  is covered by a finite number of sets in  $\mathcal{O}$ .*

**Proof.** Since  $\{x\} \times B$  is compact, we can assume at the outset that  $\mathcal{O}$  is finite, and we need only find the open set  $U$  such that  $U \times B$  is covered by  $\mathcal{O}$ .

For each  $y \in B$  the point  $(x,y)$  is in some open set  $W$  in  $\mathcal{O}$ . Since  $W$  is open, we have  $(x,y) \in U_y \times V_y \subset W$  for some open rectangle  $U_y \times V_y$ . The sets  $V_y$  cover the compact set  $B$ , so a finite number  $V_{y_1}, \dots, V_{y_k}$  also cover  $B$ . Let  $U = U_{y_1} \cap \dots \cap U_{y_k}$ . Then if  $(x',y') \in U \times B$ , we have  $y' \in V_{y_i}$  for some  $i$  (Figure 1-4), and certainly  $x' \in U_{y_i}$ . Hence  $(x',y') \in U_{y_i} \times V_{y_i}$ , which is contained in some  $W$  in  $\mathcal{O}$ .



**FIGURE 1-4**

**1-5 Corollary.** If  $A \subset \mathbf{R}^n$  and  $B \subset \mathbf{R}^m$  are compact, then  $A \times B \subset \mathbf{R}^{n+m}$  is compact.

**Proof.** If  $O$  is an open cover of  $A \times B$ , then  $O$  covers  $\{x\} \times B$  for each  $x \in A$ . By Theorem 1-4 there is an open set  $U_x$  containing  $x$  such that  $U_x \times B$  is covered by finitely many sets in  $O$ . Since  $A$  is compact, a finite number  $U_{x_1}, \dots, U_{x_n}$  of the  $U_x$  cover  $A$ . Since finitely many sets in  $O$  cover each  $U_{x_i} \times B$ , finitely many cover all of  $A \times B$ .

**1-6 Corollary.**  $A_1 \times \dots \times A_k$  is compact if each  $A_i$  is. In particular, a closed rectangle in  $\mathbf{R}^k$  is compact.

**1-7 Corollary.** A closed bounded subset of  $\mathbf{R}^n$  is compact. (The converse is also true (Problem 1-20).)

**Proof.** If  $A \subset \mathbf{R}^n$  is closed and bounded, then  $A \subset B$  for some closed rectangle  $B$ . If  $O$  is an open cover of  $A$ , then  $O$  together with  $\mathbf{R}^n - A$  is an open cover of  $B$ . Hence a finite number  $U_1, \dots, U_n$  of sets in  $O$ , together with  $\mathbf{R}^n - A$  perhaps, cover  $B$ . Then  $U_1, \dots, U_n$  cover  $A$ .

**Problems. 1-14.\*** Prove that the union of any (even infinite) number of open sets is open. Prove that the intersection of two (and hence of finitely many) open sets is open. Give a counterexample for infinitely many open sets.

**1-15.** Prove that  $\{x \in \mathbf{R}^n: |x - a| < r\}$  is open (see also Problem 1-27).

**1-16.** Find the interior, exterior, and boundary of the sets

$$\{x \in \mathbf{R}^n: |x| \leq 1\}$$

$$\{x \in \mathbf{R}^n: |x| = 1\}$$

$$\{x \in \mathbf{R}^n: \text{each } x^i \text{ is rational}\}.$$

**1-17.** Construct a set  $A \subset [0,1] \times [0,1]$  such that  $A$  contains at most one point on each horizontal and each vertical line but boundary  $A = [0,1] \times [0,1]$ . *Hint:* It suffices to ensure that  $A$  contains points in each quarter of the square  $[0,1] \times [0,1]$  and also in each sixteenth, etc.

**1-18.** If  $A \subset [0,1]$  is the union of open intervals  $(a_i, b_i)$  such that each rational number in  $(0,1)$  is contained in some  $(a_i, b_i)$ , show that boundary  $A = [0,1] - A$ .

**1-19.\*** If  $A$  is a closed set that contains every rational number  $r \in [0,1]$ , show that  $[0,1] \subset A$ .

**1-20.** Prove the converse of Corollary 1-7: A compact subset of  $\mathbf{R}^n$  is closed and bounded (see also Problem 1-28).

**1-21.\*** (a) If  $A$  is closed and  $x \notin A$ , prove that there is a number  $d > 0$  such that  $|y-x| \geq d$  for all  $y \in A$ .

(b) If  $A$  is closed,  $B$  is compact, and  $A \cap B = \emptyset$ , prove that there is  $d > 0$  such that  $|y-x| \geq d$  for all  $y \in A$  and  $x \in B$ . *Hint:* For each  $b \in B$  find an open set  $U$  containing  $b$  such that the relation holds for  $x \in U \cap B$ .

(c) Give a counterexample in  $\mathbf{R}^2$  if  $A$  and  $B$  are closed but neither is compact.

**1-22.\*** If  $U$  is open and  $C \subset U$  is compact, show that there is a compact set  $D$  such that  $C \subset \text{interior } D$  and  $D \subset U$ .

## FUNCTIONS AND CONTINUITY

A **function** from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  (sometimes called a (vector-valued) function of  $n$  variables) is a rule which associates to each point in  $\mathbf{R}^n$  some point in  $\mathbf{R}^m$ ; the point a function  $f$  associates to  $x$  is denoted  $f(x)$ . We write  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$  (read “ $f$  takes  $\mathbf{R}^n$  into  $\mathbf{R}^m$ ” or “ $f$ , taking  $\mathbf{R}^n$  into  $\mathbf{R}^m$ ,” depending on context) to indicate that  $f(x) \in \mathbf{R}^m$  is defined for  $x \in \mathbf{R}^n$ . The notation  $f: A \rightarrow \mathbf{R}^m$  indicates that  $f(x)$  is defined only for  $x$  in the set  $A$ , which is called the **domain** of  $f$ . If  $B \subset \mathbf{R}^m$ , we define  $f(B)$  as the set of all  $f(x)$  for  $x \in A$ , and if  $C \subset \mathbf{R}^m$  we define  $f^{-1}(C) = \{x \in A : f(x) \in C\}$ . The notation  $f: A \rightarrow B$  indicates that  $f(A) \subset B$ .

A convenient representation of a function  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  may be obtained by drawing a picture of its graph, the set of all 3-tuples of the form  $(x, y, f(x, y))$ , which is actually a figure in 3-space (see, e.g., [Figures 2-1](#) and [2-2](#) of Chapter 2).

If  $f, g: \mathbf{R}^n \rightarrow \mathbf{R}$ , the functions  $f + g$ ,  $f - g$ ,  $f \cdot g$ , and  $f/g$  are defined precisely as in the one-variable case. If  $f: A \rightarrow \mathbf{R}^m$  and  $g: B \rightarrow \mathbf{R}^p$ , where  $B \subset \mathbf{R}^m$ , then the **composition**  $g \circ f$  is defined by  $g \circ f(x) = g(f(x))$ ; the domain of  $g \circ f$  is  $A \cap f^{-1}(B)$ . If  $f: A \rightarrow \mathbf{R}^m$  is 1-1, that is, if  $f(x) \neq f(y)$  when  $x \neq y$ , we define  $f^{-1}: f(A) \rightarrow \mathbf{R}^n$  by the requirement that  $f^{-1}(z)$  is the unique  $x \in A$  with  $f(x) = z$ .

A function  $f: A \rightarrow \mathbf{R}^m$  determines  $m$  **component functions**  $f^1, \dots, f^m: A \rightarrow \mathbf{R}$  by  $f(x) = (f^1(x), \dots, f^m(x))$ . If conversely,  $m$  functions  $g_1, \dots, g_m: A \rightarrow \mathbf{R}$  are given, there is a unique function  $f: A \rightarrow \mathbf{R}^m$  such that  $f^i = g_i$ , namely  $f(x) = (g_1(x), \dots, g_m(x))$ . This function  $f$  will be denoted  $(g_1, \dots, g_m)$ , so that we always have  $f = (f^1, \dots, f^m)$ . If  $\pi: \mathbf{R}^n \rightarrow \mathbf{R}^n$  is the identity function,  $\pi(x) = x$ , then  $\pi^i(x) = x^i$ ; the function  $\pi^i$  is called the  **$i$ th projection function**.

The notation  $\lim_{x \rightarrow a} f(x) = b$  means, as in the one-variable case, that we can get  $f(x)$  as close to  $b$  as desired, by choosing  $x$  sufficiently close to, but not equal to,  $a$ . In mathematical terms this means that for every number  $\varepsilon > 0$  there is a number  $\delta > 0$  such that  $|f(x) - b| < \varepsilon$  for all  $x$  in the domain of  $f$  which satisfy  $0 < |x - a| < \delta$ . A function  $f: A \rightarrow \mathbf{R}^m$  is called **continuous** at  $a \in A$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ , and  $f$

simply called continuous if it is continuous at each  $a \in A$ . One of the pleasant surprises about the concept of continuity is that it can be defined without using limits. It follows from the next theorem that  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is continuous if and only if  $f^{-1}(U)$  is open whenever  $U \subset \mathbf{R}^m$  is open; if the domain of  $f$  is not all of  $\mathbf{R}^n$ , a slightly more complicated condition is needed.

**1-8 Theorem.** *If  $A \subset \mathbf{R}^n$ , a function  $f: A \rightarrow \mathbf{R}^m$  is continuous if and only if for every open set  $U \subset \mathbf{R}^m$  there is some open set  $V \subset \mathbf{R}^n$  such that  $f^{-1}(U) = V \cap A$ .*

**Proof.** Suppose  $f$  is continuous. If  $a \in f^{-1}(U)$ , then  $f(a) \in U$ . Since  $U$  is open, there is an open rectangle  $B$  with  $f(a) \in B \subset U$ . Since  $f$  is continuous at  $a$ , we can ensure that  $f(x) \in B$ , provided we choose  $x$  in some sufficiently small rectangle  $C$  containing  $a$ . Do this for each  $a \in f^{-1}(U)$  and let  $V$  be the union of all such  $C$ . Clearly  $f^{-1}(U) = V \cap A$ . The converse is similar and is left to the reader.

The following consequence of Theorem 1-8 is of great importance.

**1-9 Theorem.** *If  $f: A \rightarrow \mathbf{R}^m$  is continuous, where  $A \subset \mathbf{R}^n$ , and  $A$  is compact, then  $f(A) \subset \mathbf{R}^m$  is compact.*

**Proof.** Let  $\mathcal{O}$  be an open cover of  $f(A)$ . For each open set  $U$  in  $\mathcal{O}$  there is an open set  $V_U$  such that  $f^{-1}(U) = V_U \cap A$ . The collection of all  $V_U$  is an open cover of  $A$ . Since  $A$  is compact, a finite number  $V_{U_1}, \dots, V_{U_n}$  cover  $A$ . Then  $U_1, \dots, U_n$  cover  $f(A)$ .

If  $f: A \rightarrow \mathbf{R}$  is bounded, the extent to which  $f$  fails to be continuous at  $a \in A$  can be measured in a precise way. For  $\delta > 0$  let

$$M(a, f, \delta) = \sup\{f(x) : x \in A \text{ and } |x - a| < \delta\},$$

$$m(a, f, \delta) = \inf\{f(x) : x \in A \text{ and } |x - a| < \delta\}.$$

The **oscillation**  $o(f, a)$  of  $f$  at  $a$  is defined by  $o(f, a) = \lim_{\delta \rightarrow 0} [M(a, f, \delta) - m(a, f, \delta)]$ . This limit always exists since  $M(a, f, \delta) - m(a, f, \delta)$  decreases as  $\delta$  decreases. There are two important facts about  $o(f, a)$ .

**1-10 Theorem.** *The bounded function  $f$  is continuous at  $a$  if and only if  $o(f, a) = 0$ .*

**Proof.** Let  $f$  be continuous at  $a$ . For every number  $\varepsilon > 0$  we can choose a number  $\delta > 0$  so that  $|f(x) - f(a)| < \varepsilon$  for all  $x \in A$  with  $|x - a| < \delta$ ; thus  $M(a, f, \delta) - m(a, f, \delta) \leq 2\varepsilon$ . Since this is true for every  $\varepsilon$ , we have  $o(f, a) = 0$ . The converse is similar and is left to the reader.

**1-11 Theorem.** *Let  $A \subset \mathbf{R}^n$  be closed. If  $f: A \rightarrow \mathbf{R}$  is any bounded function, and  $\varepsilon > 0$ , then  $\{x \in A : o(f, x) \geq \varepsilon\}$  is closed.*

**Proof.** Let  $B = \{x \in A : o(f,x) \geq \varepsilon\}$ . We wish to show that  $\mathbf{R}^n - B$  is open. If  $x \in \mathbf{R}^n - B$ , then either  $x \notin A$  or else  $x \in A$  and  $o(f,x) < \varepsilon$ . In the first case, since  $A$  is closed, there is an open rectangle containing  $x$  such that  $C \subset \mathbf{R}^n - A \subset \mathbf{R}^n - B$ . In the second case there is a  $\delta > 0$  such that  $M(x,f,\delta) - m(x,f,\delta) < \varepsilon$ . Let  $C$  be an open rectangle containing  $x$  such that  $|x - y| < \delta$  for all  $y \in C$ . Then if  $y \in C$  there is a  $\delta_1$  such that  $|x - z| < \delta$  for all  $z$  satisfying  $|z - y| < \delta_1$ . Thus  $M(y,f,\delta_1) - m(y,f,\delta_1) < \varepsilon$ , and consequently  $o(y,f) < \varepsilon$ . Therefore  $C \subset \mathbf{R}^n - B$ .

**Problems. 1-23.** If  $f: A \rightarrow \mathbf{R}^m$  and  $a \in A$ , show that  $\lim_{x \rightarrow a} f(x) = b$  if and only if  $\lim_{x \rightarrow a} f^i(x) = b^i$  for  $i = 1, \dots, m$ .

**1-24.** Prove that  $f: A \rightarrow \mathbf{R}^m$  is continuous at  $a$  if and only if each  $f^i$  is.

**1-25.** Prove that a linear transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is continuous. *Hint:* Use Problem 1-10.

**1-26.** Let  $A = \{(x,y) \in \mathbf{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$ .

(a) Show that every straight line through  $(0,0)$  contains an interval around  $(0,0)$  which is  $\mathbf{R}^2 - A$ .

(b) Define  $f: \mathbf{R}^2 \rightarrow \mathbf{R}$  by  $f(x) = 0$  if  $x \notin A$  and  $f(x) = 1$  if  $x \in A$ . For  $h \in \mathbf{R}^2$  define  $g_h: \mathbf{R} \rightarrow \mathbf{R}$  by  $g_h(t) = f(th)$ . Show that each  $g_h$  is continuous at 0, but  $f$  is not continuous at  $(0,0)$ .

**1-27.** Prove that  $\{x \in \mathbf{R}^n : |x - a| < r\}$  is open by considering the function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$  with  $f(x) = |x - a|$ .

**1-28.** If  $A \subset \mathbf{R}^n$  is not closed, show that there is a continuous function  $f: A \rightarrow \mathbf{R}$  which is unbounded. *Hint:* If  $x \in \mathbf{R}^n - A$  but  $x \notin \text{interior}(\mathbf{R}^n - A)$ , let  $f(y) = 1/|y - x|$ .

**1-29.** If  $A$  is compact, prove that every continuous function  $f: A \rightarrow \mathbf{R}$  takes on a maximum and a minimum value.

**1-30.** Let  $f: [a,b] \rightarrow \mathbf{R}$  be an increasing function. If  $x_1, \dots, x_n \in [a,b]$  are distinct, show that  $\sum_{i=1}^n o(f, x_i) < f(b) - f(a)$ .

## *Differentiation*

### **BASIC DEFINITIONS**

Recall that a function  $f: \mathbf{R} \rightarrow \mathbf{R}$  is differentiable at  $a \in \mathbf{R}$  if there is a number  $f'(a)$  such that

$$(1) \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

This equation certainly makes no sense in the general case of a function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ , but can be reformulated in a way that does. If  $\lambda: \mathbf{R} \rightarrow \mathbf{R}$  is the linear transformation defined by  $\lambda(h) = f'(a) \cdot h$ , then equation (1) is equivalent to

$$(2) \lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0.$$

Equation (2) is often interpreted as saying that  $\lambda + f(a)$  is a good approximation to  $f$  at  $a$  (see Problem 2-9). Henceforth we focus our attention on the linear transformation  $\lambda$  and reformulate the definition of differentiability as follows.

A function  $f: \mathbf{R} \rightarrow \mathbf{R}$  is differentiable at  $a \in \mathbf{R}$  if there is a linear transformation  $\lambda: \mathbf{R} \rightarrow \mathbf{R}$  such that

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - \lambda(h)}{h} = 0.$$

In this form the definition has a simple generalization to higher dimensions:

A function  $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$  is **differentiable** at  $a \in \mathbf{R}^n$  if there is a linear transformation  $\lambda: \mathbf{R}^n \rightarrow \mathbf{R}^m$  such that

$$\lim_{h \rightarrow 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Note that  $h$  is a point of  $\mathbf{R}^n$  and  $f(a+h) - f(a) - \lambda(h)$  a point of  $\mathbf{R}^m$ , so the norm signs are essential. The linear transformation  $\lambda$  is denoted  $Df(a)$  and called the **derivative** of  $f$  at  $a$ . The justification for the phrase “the linear transformation  $\lambda$ ” is



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