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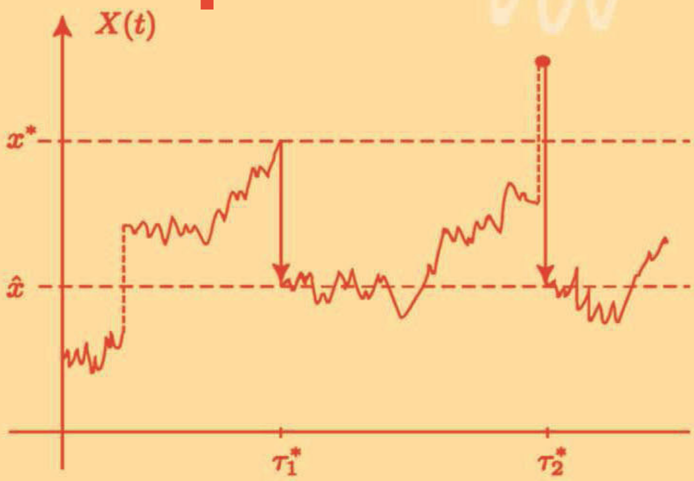
Applied Stochastic Control of Jump Diffusions

Second Edition



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Applied Stochastic Control of Jump Diffusions

2nd Edition

With 27 Figures

Springer

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To my family

Eva, Elise, Anders, and Karina

B. Ø.

A tous ceux qui m'accompagnent

A. S.

Preface to the Second Edition

In this second edition, we have added a chapter on optimal control of random jump fields (solutions of stochastic *partial* differential equations) and partial information control (Chap. 10). We have also added a section on optimal stopping with delayed information (Sect. 2.3). It has always been our intention to give a contemporary presentation of applied stochastic control, and we hope that the addition of these recent developments will contribute in this direction.

We have also made a number of corrections and other improvements, many of them based on helpful comments from our readers. In particular, we would like to thank Andreas Kyprianou for his valuable communications. We are also grateful to (in alphabetical order) Knut Aase, Jean-Philippe Chancelier, Inga Eide, Emil Framnes, Arne-Christian Lund, Jose-Luis Menaldi, Tamás K. Papp, Atle Seierstad, and Jens Arne Sukkestad for pointing out errors and suggesting improvements. Our special thanks go to Martine Verneuille for her skillful typing.

Oslo and Paris, November 2006 Bernt Øksendal and Agnès Sulem

Preface of the First Edition

Jump diffusions are solutions of stochastic differential equations driven by Lévy processes. Since a Lévy process $\eta(t)$ can be written as a linear combination of t, a Brownian motion $B(t)$ and a pure jump process, jump diffusions represent a natural and useful generalization of Itô diffusions. They have received a lot of attention in the last years because of their many applications, particularly in economics.

There exist today several excellent monographs on Lévy processes. However, very few of them – if any – discuss the optimal control, optimal stopping, and impulse control of the corresponding jump diffusions, which is the subject of this book. Moreover, our presentation differs from these books in that it emphasizes the applied aspect of the theory. Therefore, we focus mostly on useful verification theorems and we illustrate the use of the theory by giving examples and exercises throughout the text. Detailed solutions of some of the exercises are given at the end of the book. The exercises to which a solution is provided, are marked with an asterix $*$. It is our hope that this book will fill a gap in the literature and that it will be a useful text for students, researchers, and practitioners in stochastic analysis and its many applications. Although most of our results are motivated by examples in economics and finance, the results are general and can be applied in a wide variety of situations. To emphasize this, we have also included examples in biology and physics/engineering.

This book is partially based on courses given at the Norwegian School of Economics and Business Administration (NHH) in Bergen, Norway, during the Spring semesters 2000 and 2002, at INSEA in Rabat, Morocco in September 2000, at Odense University in August 2001 and at ENSAE in Paris in February 2002.

Oslo and Paris, August 2004 Bernt Øksendal and Agnès Sulem

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Oslo and Paris, August 2004 Bernt Øksendal and Agnès Sulem

Contents

1

Stochastic Calculus with Jump Diffusions

1.1 Basic Definitions and Results on Lévy Processes

In this chapter we present the basic concepts and results needed for the applied calculus of jump diffusions. Since there are several excellent books which give a detailed account of this basic theory, we will just briefly review it here and refer the reader to these books for more information.

Definition 1.1. *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. An $\mathcal{F}_t$-adapted process $\{\eta(t)\}_{t\geq 0} = \{\eta_t\}_{t\geq 0} \subset \mathbb{R}$ with $\eta_0 = 0$ a.s. is called a* Lévy process *if η_t is continuous in probability and has* stationary and independent *increments.*

Theorem 1.2. *Let $\{\eta_t\}$ be a Lévy process. Then η_t has a* càdlàg *version (right continuous with left limits) which is also a Lévy process.*

Proof. See, e.g., [P, S]. □

In view of this result we will from now on assume that the Lévy processes we work with are càdlàg.

The *jump* of η_t at $t \geq 0$ is defined by

$$\Delta\eta_t = \eta_t - \eta_{t^-}. \tag{1.1.1}$$

Let $\mathbf{B}_0$ be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\bar{U}$ does not contain 0. For $U \in \mathbf{B}_0$ we define

$$N(t,U) = N(t,U,\omega) = \sum_{s:0<s\leq t} \mathcal{X}_U(\Delta\eta_s). \tag{1.1.2}$$

In other words, $N(t,U)$ is the number of jumps of size $\Delta\eta_s \in U$ which occur before or at time t. $N(t,U)$ is called the *Poisson random measure (or jump measure)* of $\eta(\cdot)$.

Remark 1.3. Note that $N(t,U)$ is *finite* for all $U \in \mathbf{B}_0$. To see this we proceed as follows: Define

$$T_1(\omega) = \inf\{t > 0; \eta_t \in U\}.$$

We claim that $T_1(\omega) > 0$ a.s. To prove this note that by right continuity of paths we have

$$\lim_{t \to 0^+} \eta(t) = \eta(0) = 0 \quad \text{a.s.}$$

Therefore, for all $\varepsilon > 0$ there exists $t(\varepsilon) > 0$ such that $|\eta(t)| < \varepsilon$ for all $t < t(\varepsilon)$. This implies that $\eta(t) \notin U$ for all $t < t(\varepsilon)$, if $\varepsilon < \mathrm{dist}(0,U)$.

Next define inductively

$$T_{n+1}(\omega) = \inf\{t > T_n(\omega); \Delta\eta_t \in U\}.$$

Then by the above argument $T_{n+1} > T_n$ a.s. We claim that

$$T_n \to \infty \quad \text{as} \quad n \to \infty, \text{a.s.}$$

Assume not. Then $T_n \to T < \infty$. But then

$$\lim_{s \to T^-} \eta(s) \quad \text{cannot exist,}$$

contradicting the existence of left limits of the paths.

It is well known that Brownian motion $\{B(t)\}_{t \geq 0}$ has stationary and independent increments. Thus $B(t)$ is a Lévy process. Another important example is the following.

Example 1.4 (The Poisson Process). The Poisson process $\pi(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ and such that

$$P[\pi(t) = n] = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}; \quad n = 0, 1, 2, \ldots$$

Theorem 1.5. *[P, Theorem 1.35].*

1. *The set function $U \to N(t,U,\omega)$ defines a σ-finite measure on $\mathbf{B}_0$ for each fixed t, ω. The differential form of this measure is written $N(t, \mathrm{d}z)$.*
2. *The set function $[a,b) \times U \to N(b,U,\omega) - N(a,U,\omega); [a,b) \subset [0,\infty), U \in \mathbf{B}_0$ defines a σ-finite measure for each fixed ω. The differential form of this measure is written $N(\mathrm{d}t, \mathrm{d}z)$.*
3. *The set function*
$$\nu(U) = E[N(1,U)], \tag{1.1.3}$$
where $E = E_P$ denotes expectation with respect to P, also defines a σ-finite measure on $\mathbf{B}_0$, called the Lévy measure *of $\{\eta_t\}$.*
4. *Fix $U \in \mathbf{B}_0$. Then the process*
$$\pi_U(t) := \pi_U(t,\omega) := N(t,U,\omega)$$
is a Poisson process *of* intensity *$\lambda = \nu(U)$.*

Example 1.6 (The Compound Poisson Process). Let $X(n)$, $n \in \mathbb{N}$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}$ with common distribution $\mu_{X(1)} = \mu_X$ and let $\pi(t)$ be a Poisson process of intensity λ, independent of all the $X(n)$s.

The *compound Poisson process* $Y(t)$ is defined by

$$Y(t) = X(1) + \cdots + X(\pi(t)), \quad t \geq 0. \tag{1.1.4}$$

An increment of this process is given by

$$Y(s) - Y(t) = \sum_{k=\pi(t)+1}^{\pi(s)} X(k), \quad s > t.$$

This is independent of $X(1), \ldots, X(\pi(t))$, and its distribution depends only on the difference $(s - t)$ and on the distribution of $X(1)$. Thus $Y(t)$ is a Lévy process.

To find the Lévy measure ν of $Y(t)$ note that if $U \in \mathbf{B}_0$ then

$$\begin{aligned}\nu(U) = E[N(1,U)] = E\left[\sum_{s;0<s\leq 1} \mathcal{X}_U(\Delta Y(s))\right] \\ = E[(\text{number of jumps}) \times X_U(\text{jump})] = E[\pi(1)\mathcal{X}_U(X)] = \lambda\mu_X(U),\end{aligned}$$

by independence. We conclude that

$$\nu = \lambda\mu_X. \tag{1.1.5}$$

This shows that a Lévy process can be represented by a compound Poisson process if and only if its Lévy measure is finite. Note, however, that there are many interesting Lévy processes η_t with infinite Lévy measure ν, in fact even with $\int_{|z|<1} |z|\nu(\mathrm{d}z) = \infty$. See, e.g., [B]. In general, one can prove that for any fixed $R > 0$ the processes

$$M_t^{(k)} := \int_{1/k \leq |z| \leq R} z(N(t, \mathrm{d}z) - t\nu(\mathrm{d}z)), \quad k = 1, 2, \ldots$$

are $L^2(P)$-martingales and they converge in $L^2(P)$ to a martingale M_t denoted by

$$M_t = \int_{|z| \leq R} z(N(t, \mathrm{d}z) - t\nu(\mathrm{d}z)).$$

In fact we have

Theorem 1.7 (Itô–Lévy Decomposition [JS]). *Let $\{\eta_t\}$ be a Lévy process. Then η_t has the decomposition*

$$\eta_t = \alpha t + \sigma B(t) + \int_{|z|<R} z\widetilde{N}(t, \mathrm{d}z) + \int_{|z| \geq R} zN(t, \mathrm{d}z), \tag{1.1.6}$$

for some constants $\alpha \in \mathbb{R}$, $\sigma \in \mathbb{R}$, $R \in [0, \infty]$. *Here*

$$\widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t \tag{1.1.7}$$

is the compensated Poisson random measure *of* $\eta(\cdot)$, *and* $B(t)$ *is a Brownian motion independent of* $\tilde{N}(\mathrm{d}t, \mathrm{d}z)$. *For each* $A \in \mathbf{B}_0$ *the process*

$$M_t := \widetilde{N}(t, A) \quad \textit{is a} \text{ martingale.} \tag{1.1.8}$$

If $\alpha = 0$ *and* $R = \infty$, *we call* η_t *a* Lévy martingale*.*

Theorem 1.8. *We can always choose* $R = 1$. *If*

$$E|\eta_t| < \infty \quad \textit{for all}\, t \geq 0,$$

then

$$\int_{|z|\geq 1} |z|\nu(\mathrm{d}z) < \infty$$

and hence we may choose $R = \infty$ *and write*

$$\eta_t = \alpha t + \sigma B(t) + \int_{\mathbb{R}} z\widetilde{N}(t, \mathrm{d}z).$$

(See [S, Theorem 25.3].)

Theorem 1.9 ([P]). *A Lévy process is a strong Markov process.*

Theorem 1.10 (The Lévy–Khintchine formula [P]). *Let* $\{\eta_t\}$ *be a Lévy process with Lévy measure* ν. *Then* $\int_{\mathbb{R}} \min(1, z^2)\nu(\mathrm{d}z) < \infty$ *and*

$$E[\mathrm{e}^{\mathrm{i}u\eta_t}] = \mathrm{e}^{t\psi(u)}, \quad u \in \mathbb{R}, \tag{1.1.9}$$

where

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + \mathrm{i}\alpha u + \int_{|z|<R} \{\mathrm{e}^{\mathrm{i}uz} - 1 - \mathrm{i}uz\}\nu(\mathrm{d}z) + \int_{|z|\geq R} (\mathrm{e}^{\mathrm{i}uz} - 1)\nu(\mathrm{d}z). \tag{1.1.10}$$

Conversely, given constants α, σ^2, *and a measure* ν *on* $\mathbf{B}_0$ *s.t.*

$$\int_{\mathbb{R}} \min(1, z^2)\nu(\mathrm{d}z) < \infty,$$

there exists a Lévy process $\eta(t)$ *(unique in law) such that (1.1.9–1.1.10) hold.*

Note. It is possible that $\displaystyle\int_{|z|\leq R} |z|\nu(\mathrm{d}z) = \infty$.

Theorem 1.11. *[P, Corollary p. 48]. A Lévy process is a* semimartingale.

Definition 1.12. *[P]. Let* $\mathbf{D}_{\text{ucp}}$ *denote the space of càdlàg adapted processes, equipped with the topology of uniform convergence on compacts in probability* (ucp) : $H_n \to H$ ucp *if for all* $t > 0 \sup_{0 \le s \le t} |H_n(s) - H(s)| \to 0$ *in probability* ($A_n \to A$ *in probability if for all* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that* $n \ge n_\varepsilon \Rightarrow$ Prob.$(|A_n - A| > \varepsilon) < \varepsilon$*).*

Let $\mathbf{L}_{\text{ucp}}$ *denote the space of adapted* càglàd *processes (left continuous with right limits), equipped with the* ucp *topology. If* $H(t)$ *is a step function of the form*

$$H(t) = H_0 \mathcal{X}_{\{0\}}(t) + \sum_i H_i \mathcal{X}_{(T_i, T_{i+1}]}(t),$$

where $H_i \in \mathcal{F}_{T_i}$ *and* $0 = T_0 \le T_1 \le \cdots \le T_{n+1} < \infty$ *are* $\mathcal{F}_t$*-stopping times and* X *is càdlàg, we define*

$$J_X H(t) := \int_0^t H_s \mathrm{d}X_s := H_0 X_0 + \sum_i H_i (X_{T_{i+1} \wedge t} - X_{T_i \wedge t}), \quad t \ge 0.$$

Theorem 1.13. *[P, p. 51]. Let* X *be a semimartingale. Then the mapping* J_X *can be extended to a continuous linear map*

$$J_X : \mathbf{L}_{\text{ucp}} \to \mathbf{D}_{\text{ucp}}.$$

This construction allows us to define stochastic integrals of the form

$$\int_0^t H(s) \mathrm{d}\eta_s$$

for all $H \in \mathbf{L}_{\text{ucp}}$. (See also Remark 1.18.) In view of the decomposition (1.1.6) this integral can be split into integrals with respect to $\mathrm{d}s$, $\mathrm{d}B(s)$, $\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$, and $N(\mathrm{d}s, \mathrm{d}z)$. This makes it natural to consider the more general stochastic integrals of the form

$$X(t) = X(0) + \int_0^t \alpha(s,\omega)\mathrm{d}s + \int_0^t \beta(s,\omega)\mathrm{d}B(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\bar{N}(\mathrm{d}s, \mathrm{d}z), \tag{1.1.11}$$

where the integrands are satisfying the appropriate conditions for the integrals to exist and we for simplicity have put

$$\bar{N}(\mathrm{d}s, \mathrm{d}z) = \begin{cases} N(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}s & \text{if } |z| < R \\ N(\mathrm{d}s, \mathrm{d}z) & \text{if } |z| \ge R, \end{cases}$$

with R as in Theorem 1.7. As is customary we will use the following shorthand differential notation for processes $X(t)$ satisfying (1.1.11):

$$\mathrm{d}X(t) = \alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(\mathrm{d}t, \mathrm{d}z). \tag{1.1.12}$$

We call such processes *Itô–Lévy processes.*

Recall that a semimartingale $M(t)$ is called a *local* martingale up to time T (with respect to P) if there exists an increasing sequence of $\mathcal{F}_t$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = T$ a.s. and

$$M(t \wedge \tau_n) \quad \text{is a martingale with respect to } P \text{ for all } n.$$

Note that

1. If
$$E\left[\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(\mathrm{d}z)\mathrm{d}t\right] < \infty, \tag{1.1.13}$$
then the process
$$M(t) := \int_0^t \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad 0 \le t \le T$$
is a *martingale*.

2. If
$$\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(\mathrm{d}z)\mathrm{d}t < \infty \text{ a.s.}, \tag{1.1.14}$$
then $M(t)$ is a *local* martingale, $0 \le t \le T$.

1.2 The Itô Formula and Related Results

We now come to the important Itô formula for Itô–Lévy processes:

If $X(t)$ is given by (1.1.12) and $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function, is the process $Y(t) := f(t, X(t))$ again an Itô–Lévy process and if so, how do we represent it in the form (1.1.12)?

If we argue heuristically and use our knowledge of the classical Itô formula it is easy to guess what the answer is:

Let $X^{(c)}(t)$ be the continuous part of $X(t)$, i.e., $X^{(c)}(t)$ is obtained by removing the jumps from $X(t)$. Then an increment in $Y(t)$ stems from an increment in $X^{(c)}(t)$ plus the jumps (coming from $N(\cdot,\cdot)$). Hence in view of the classical Itô formula we would guess that

$$\begin{aligned}\mathrm{d}Y(t) &= \frac{\partial f}{\partial t}(t,X(t))\mathrm{d}t + \frac{\partial f}{\partial x}(t,X(t))\mathrm{d}X^{(c)}(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t)) \cdot \beta^2(t)\mathrm{d}t \\ &\quad + \int_{\mathbb{R}} \{f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-))\} N(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

It can be proved that our guess is correct. Since

$$\mathrm{d}X^{(c)}(t) = \left(\alpha(t) - \int_{|z|<R} \gamma(t,z)\nu(\mathrm{d}z)\right)\mathrm{d}t + \beta(t)\mathrm{d}B(t),$$

this gives the following result.

Theorem 1.14 (The One-Dimensional Itô Formula [BL, A, P]). *Suppose $X(t) \in \mathbb{R}$ is an Itô–Lévy process of the form*

$$\mathrm{d}X(t) = \alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,z,\omega)\bar{N}(\mathrm{d}t,\mathrm{d}z), \tag{1.2.1}$$

where

$$\bar{N}(\mathrm{d}t,\mathrm{d}z) = \begin{cases} N(\mathrm{d}t,\mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t & \text{if} \quad |z| < R \\ N(\mathrm{d}t,\mathrm{d}z) & \text{if} \quad |z| \geq R \end{cases} \tag{1.2.2}$$

for some $R \in [0,\infty]$.

Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t,X(t))$. Then $Y(t)$ is again an Itô–Lévy process and

$$\begin{aligned} \mathrm{d}Y(t) = {} & \frac{\partial f}{\partial t}(t,X(t))\mathrm{d}t + \frac{\partial f}{\partial x}(t,X(t))[\alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t)] \\ & + \frac{1}{2}\beta^2(t,\omega)\frac{\partial^2 f}{\partial x^2}(t,X(t))\mathrm{d}t \\ & + \int_{|z|<R} \Big\{ f(t,X(t^-)+\gamma(t,z)) - f(t,X(t^-)) \\ & \qquad - \frac{\partial f}{\partial x}(t,X(t^-))\gamma(t,z) \Big\} \nu(\mathrm{d}z)\mathrm{d}t \\ & + \int_{\mathbb{R}} \{ f(t,X(t^-)+\gamma(t,z)) - f(t,X(t^-)) \} \bar{N}(\mathrm{d}t,\mathrm{d}z). \end{aligned} \tag{1.2.3}$$

Note. If $R = 0$ then $\bar{N} = N$ everywhere.
If $R = \infty$ then $\bar{N} = \widetilde{N}$ everywhere.

Example 1.15 (The Geometric Lévy Process). Consider the stochastic differential equation (SDE)

$$\mathrm{d}\,X(t) = X(t^-)\left[\alpha\,\mathrm{d}t + \beta\,\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(\mathrm{d}t,\mathrm{d}z)\right], \tag{1.2.4}$$

where α, β are constants and $\gamma(t,z) \geq -1$. To find the solution $X(t)$ of this equation we rewrite it as follows:

$$\frac{\mathrm{d}X(t)}{X(t^-)} = \alpha\,\mathrm{d}t + \beta\,\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(\mathrm{d}t,\mathrm{d}z).$$

Now define

$$Y(t) = \ln X(t).$$

Then by Itô's formula,

$$\begin{aligned}
\mathrm{d}Y(t) &= \frac{X(t)}{X(t)}[\alpha\,\mathrm{d}t + \beta\,\mathrm{d}B(t)] - \frac{1}{2}\beta^2 X^{-2}(t)X^2(t)\mathrm{d}t \\
&\quad + \int_{|z|<R} \{\ln(X(t^-) + \gamma(t,z)X(t^-)) - \ln(X(t^-)) \\
&\quad - X^{-1}(t^-)\gamma(t,z)X(t^-)\}\nu(\mathrm{d}z)\mathrm{d}t \\
&\quad + \int_{\mathbb{R}} \{\ln(X(t^-) + \gamma(t,z)X(t^-)) - \ln(X(t^-))\}\bar{N}(\mathrm{d}t,\mathrm{d}z) \\
&= \left(\alpha - \frac{1}{2}\beta^2\right)\mathrm{d}t + \beta\,\mathrm{d}B(t) + \int_{|z|<R} \{\ln(1+\gamma(t,z)) - \gamma(t,z)\}\nu(\mathrm{d}z)\mathrm{d}t \\
&\quad + \int_{\mathbb{R}} \ln(1+\gamma(t,z))\bar{N}(\mathrm{d}t,\mathrm{d}z).
\end{aligned}$$

Hence

$$\begin{aligned}
Y(t) = Y(0) + \left(\alpha - \frac{1}{2}\beta^2\right)t + \beta B(t) + \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,z)) \\
- \gamma(s,z)\}\nu(\mathrm{d}z)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\bar{N}(\mathrm{d}s,\mathrm{d}z)
\end{aligned}$$

and this gives the solution

$$\begin{aligned}
X(t) = X(0)\exp\Bigg\{ &\left(\alpha - \frac{1}{2}\beta^2\right)t + \beta B(t) \\
&+ \int_0^t \int_{|z|<R} \{\ln(1+\gamma(s,z)) - \gamma(s,z)\}\nu(\mathrm{d}z)\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\bar{N}(\mathrm{d}s,\mathrm{d}z)\Bigg\}. \qquad (1.2.5)
\end{aligned}$$

In analogy with the diffusion case ($N = 0$) we call this process $X(t)$ a *geometric Lévy process*. It is often used as a model for stock prices. See, e.g., [B].

Next we formulate the corresponding multidimensional version of Theorem 1.14.

Theorem 1.16 (The Multidimensional Itô Formula). *Let $X(t) \in \mathbb{R}^n$ be an Itô–Lévy process of the form*

$$\mathrm{d}X(t) = \alpha(t,\omega)\mathrm{d}t + \sigma(t,\omega)\mathrm{d}B(t) + \int_{\mathbb{R}^\ell} \gamma(t,z,\omega)\bar{N}(\mathrm{d}t,\mathrm{d}z), \qquad (1.2.6)$$

where $\alpha : [0,T] \times \Omega \to \mathbb{R}^n$, $\sigma : [0,T] \times \Omega \to \mathbb{R}^{n\times m}$, and $\gamma : [0,T] \times \mathbb{R}^\ell \times \Omega \to \mathbb{R}^{n\times\ell}$ are adapted processes such that the integrals exist. Here $B(t)$ is an m-dimensional Brownian motion and

$$
\begin{aligned}
\bar{N}(\mathrm{d}t, \mathrm{d}z)^T &= (\bar{N}_1(\mathrm{d}t, \mathrm{d}z_1), \ldots, \bar{N}_\ell(\mathrm{d}t, \mathrm{d}z_\ell)) \\
&= (N_1(\mathrm{d}t, \mathrm{d}z_1) - \mathcal{X}_{|z_1|<R_1}\nu_1(\mathrm{d}z_1)\mathrm{d}t, \ldots, N_\ell(\mathrm{d}t, \mathrm{d}z_\ell) \\
&\quad - \mathcal{X}_{|z_\ell|<R_\ell}\nu_\ell(\mathrm{d}z_\ell)\mathrm{d}t),
\end{aligned}
$$

where $\{N_j\}$ are independent Poisson random measures with Lévy measures ν_j coming from ℓ independent (one-dimensional) Lévy processes $\eta_1, \ldots, \eta_\ell$.

Note that each column $\gamma^{(k)}$ of the $n \times \ell$ matrix $\gamma = [\gamma_{ij}]$ depends on z only through the kth coordinate z_k, i.e.,

$$\gamma^{(k)}(t, z, \omega) = \gamma^{(k)}(t, z_k, \omega), \quad z = (z_1, \ldots, z_\ell) \in \mathbb{R}^\ell.$$

Thus the integral on the right of (1.2.6) *is just a shorthand matrix notation. When written out in detail component number i of $X(t)$ in* (1.2.6)*, $X_i(t)$ gets the form*

$$
\begin{aligned}
\mathrm{d}X_i(t) &= \alpha_i(t,\omega)\mathrm{d}t + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)\mathrm{d}B_j(t) \\
&\quad + \sum_{j=1}^{\ell} \int_{\mathbb{R}} \gamma_{ij}(t, z_j, \omega)\bar{N}_j(\mathrm{d}t, \mathrm{d}z_j), \quad 1 \le i \le n. \qquad (1.2.7)
\end{aligned}
$$

Let $f \in C^{1,2}([0,T]) \times \mathbb{R}^n; \mathbb{R})$. Put $Y(t) = f(t, X(t))$. Then

$$
\begin{aligned}
\mathrm{d}Y(t) &= \frac{\partial f}{\partial t}\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\alpha_i \mathrm{d}t + \sigma_i \mathrm{d}B(t)) + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}\mathrm{d}t \\
&\quad + \sum_{k=1}^{\ell} \int_{|z_k|<R_k} \Big\{ f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-)) \\
&\qquad - \sum_{i=1}^{n} \gamma_i^{(k)}(t, z_k) \frac{\partial f}{\partial x_i}(X(t^-)) \Big\} \nu_k(\mathrm{d}z_k)\mathrm{d}t \\
&\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} \{ f(t, X(t^-) + \gamma^{(k)}(t, z_k)) - f(t, X(t^-)) \} \bar{N}_k(\mathrm{d}t, \mathrm{d}z_k),
\end{aligned}
\qquad (1.2.8)
$$

where $\gamma^{(k)} \in \mathbb{R}^n$ is column number k of the $n \times \ell$ matrix $\gamma = [\gamma_{ik}]$ and $\gamma_i^{(k)} = \gamma_{ik}$ is the coordinate number i of $\gamma^{(k)}$.

Theorem 1.17 (The Itô–Lévy Isometry). *Let $X(t) \in \mathbb{R}^n$ be as in (1.2.6) but with $X(0) = 0$ and $\alpha = 0$. Then*

$$E[X^2(T)] = E\left[\int_0^T \left\{\sum_{i=1}^n \sum_{j=1}^m \sigma_{ij}^2(t) + \sum_{i=1}^n \sum_{j=1}^\ell \int_{\mathbb{R}} \gamma_{ij}^2(t,z_j)\nu_j(\mathrm{d}z_j)\right\}\mathrm{d}t\right]$$
$$= \sum_{i=1}^n E\left[\int_0^T \left\{\sum_{j=1}^m \sigma_{ij}^2(t) + \sum_{j=1}^\ell \int_{\mathbb{R}} \gamma_{ij}^2(t,z_j)\nu_j(\mathrm{d}z_j)\right\}\mathrm{d}t\right] \quad (1.2.9)$$

provided that the right-hand side is finite.

Proof. This follows from the Itô formula applied to $f(t,x) = x^2 = |x|^2$. We omit the details. □

Remark 1.18. As a special case of Theorem 1.17 assume that

$$\mathrm{d}X(t) = \mathrm{d}\eta(t) = \int_{\mathbb{R}} z\widetilde{N}(\mathrm{d}t,\mathrm{d}z) \in \mathbb{R}$$

with $E[X^2(T)] = T\int_{\mathbb{R}} z^2\nu(\mathrm{d}z) < \infty$. Then we get the isometry

$$E\left[\left(\int_0^T H(t)\mathrm{d}\eta(t)\right)^2\right] = E\left[\int_0^T H^2(t)\mathrm{d}t\right]\int_{\mathbb{R}} z^2\nu(\mathrm{d}z)$$

for all $H \in \mathbf{L}_{\mathrm{ucp}}$ (see Definition 1.12) such that $H \in L^2([0,T]\times\Omega)$, i.e., such that

$$\|H\|^2_{L^2([0,T]\times\Omega)} := E\left[\int_0^T H^2(t)\mathrm{d}t\right] < \infty.$$

Using this we can in the usual way extend the definition of the integral

$$\int_0^T Y(t)\mathrm{d}\eta(t) \in L^2(\Omega)$$

to all processes $Y(t)$ which are limits in $L^2([0,T]\times\Omega)$ of processes $H_n(t) \in \mathbf{L}_{\mathrm{ucp}} \cap L^2([0,T]\times\Omega)$. We will call such processes $Y(t)$ *predictable processes.*

1.3 Lévy Stochastic Differential Equations

The geometric Lévy process is an example of a *Lévy diffusion* , i.e., the solution of a SDE driven by Lévy processes.

Theorem 1.19 (Existence and Uniqueness of Solutions of Lévy SDEs). *Consider the following Lévy SDE in* $\mathbb{R}^n$*:* $X(0) = x_0 \in \mathbb{R}^n$ *and*

$$\mathrm{d}X(t) = \alpha(t,X(t))\mathrm{d}t + \sigma(t,X(t))\mathrm{d}B(t) + \int_{\mathbb{R}^n} \gamma(t,X(t^-),z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \quad (1.3.1)$$

where $\alpha : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ *and* $\gamma : [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n\times \ell}$ *satisfy the following conditions*

(At most linear growth) There exists a constant $C_1 < \infty$ *such that*

$$\|\sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{\ell} |\gamma_k(t,x,z)|^2 \nu_k(\mathrm{d}z_k) \le C_1(1+|x|^2)$$

for all $x \in \mathbb{R}^n$

(Lipschitz continuity) There exists a constant $C_2 < \infty$ *such that*

$$\begin{aligned}&\|\sigma(t,x)-\sigma(t,y)\|^2 + |\alpha(t,x)-\alpha(t,y)|^2 \\ &\quad + \sum_{k=1}^{\ell}\int_{\mathbb{R}} |\gamma^{(k)}(t,x,z_k) - \gamma^{(k)}(t,y,z_k)|^2 \nu_k(\mathrm{d}z_k) \le C_2|x-y|^2,\end{aligned}$$

for all $x, y \in \mathbb{R}^n$.

Then there exists a unique càdlàg adapted solution $X(t)$ *such that*

$$E[|X(t)|^2] < \infty \quad \textit{for all } t.$$

Solutions of Lévy SDEs in the *time homogeneous* case, i.e., when $\alpha(t,x) = \alpha(x)$, $\sigma(t,x) = \sigma(x)$, and $\gamma(t,x,z) = \gamma(x,z)$ are called *jump diffusions* (or *Lévy diffusions*).

Theorem 1.20. *A jump diffusion is a strong Markov process.*

Proof. See [P, Theorem V.32]. □

Definition 1.21. *Let* $X(t) \in \mathbb{R}^n$ *be a jump diffusion. Then the generator* A *of* X *is defined on functions* $f : \mathbb{R}^n \to \mathbb{R}$ *by*

$$Af(x) = \lim_{t\to 0^+} \frac{1}{t}\left\{E^x[f(X(t))] - f(x)\right\} \quad \textit{(if the limit exists)},$$

where $E^x[f(X(t))] = E[f(X^{(x)}(t))]$, $X^{(x)}(0) = x$.

Theorem 1.22. *Suppose* $f \in C_0^2(\mathbb{R}^n)$. *Then* $Af(x)$ *exists and is given by*

$$\begin{aligned}Af(x) &= \sum_{i=1}^{n} \alpha_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &\quad + \int_{\mathbb{R}} \sum_{k=1}^{\ell}\{f(x+\gamma^{(k)}(x,z)) - f(x) - \nabla f(x)\cdot\gamma^{(k)}(x,z)\}\nu_k(\mathrm{d}z_k).\end{aligned} \tag{1.3.2}$$

From now on we *define* $Af(x)$ by the expression (1.3.2) for all f such that the partial derivatives of f and the integrals in (1.3.2) exist at x.

Theorem 1.23 (The Dynkin Formula I). *Let $X(t) \in \mathbb{R}^n$ be a jump diffusion and let $f \in C_0^2(\mathbb{R}^n)$. Let τ be a stopping time such that*

$$E^x[\tau] < \infty.$$

Then

$$E^x[f(X(\tau))] = f(x) + E^x\left[\int_0^\tau Af(X(s))\mathrm{d}s\right].$$

Proof. This follows by combining the Itô formula (1.2.8) with the formula (1.3.2) for A and taking expectation. □

This version is usually strong enough for applications in the case when there are no jumps ($N = 0$). However, for jump diffusions we need the following stronger, localized version.

Theorem 1.24 (The Dynkin Formula II). *Let $X(t) \in \mathbb{R}^n$ be a jump diffusion, $G \subset \mathbb{R}^n$ be an open set and let $f \in C^2(G) \cap C(\bar{G})$. Let $\tau < \infty$ be a stopping time. Suppose that*

$$\tau \le \tau_G := \inf\{t > 0; X(t) \notin G\} \tag{1.3.3}$$

$$X(\tau) \in \bar{G} \quad a.s. \tag{1.3.4}$$

$$E^x\left[|f(X(\tau))| + \int_0^\tau |Af(X(t))|\mathrm{d}t\right] < \infty. \tag{1.3.5}$$

Then we have

$$E^x[f(X(\tau))] = f(x) + E^x\left[\int_0^\tau (Af)(X(t))\mathrm{d}t\right].$$

Definition 1.25. *In general, if $\{\psi_m\}_{m=1}^\infty$ and g are functions defined on a set $G \subset \mathbb{R}^n$, we say that $\psi_m \to g$* pointwise dominatedly *in G if $\psi_m(x) \to g(x)$ for all $x \in G$ and there exists a constant $C < \infty$ such that*

$$|\psi_m(x)| \le C|g(x)| \quad \textit{for all } x \in G,\ m = 1, 2, \ldots$$

Proof of Theorem 1.24. Choose $f_m \in C_0^2(\mathbb{R}^n)$ such that $f_m \to f$ pointwise dominatedly in $\bar{G}$ and $\frac{\partial f_m}{\partial x_i} \to \frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f_m}{\partial x_i \partial x_j} \to \frac{\partial^2 f}{\partial x_i \partial x_j}$, and $Af_m \to Af$ pointwise dominatedly in G for all $i, j = 1, \ldots, n$. Then apply Theorem 1.23 to each f_m and $\tau \wedge k$, $k = 1, 2, \ldots$ Let $m, k \to \infty$ and apply the dominated convergence theorem. □

1.4 The Girsanov Theorem and Applications

The Girsanov theorem and the related concept of an equivalent local martingale measure (ELMM) are important in the applications of stochastic analysis to finance. In this chapter we first give a general semimartingale discussion and then we apply it to Itô–Lévy processes. We refer to [Ka] for more details.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space. Let Q be another probability measure on $\mathcal{F}_T$. We say that Q is *equivalent* to $P \mid \mathcal{F}_T$ if $P \mid \mathcal{F}_T \ll Q$ and $Q \ll P \mid \mathcal{F}_T$, or, equivalently, if P and Q have the same zero sets in $\mathcal{F}_T$. By the Radon–Nikodym theorem this is the case if and only if we have

$$\mathrm{d}Q(\omega) = Z(T)\mathrm{d}P(\omega) \quad \text{and} \quad \mathrm{d}P(\omega) = Z^{-1}(T)\mathrm{d}Q(\omega) \quad \text{on } \mathcal{F}_T$$

for some $\mathcal{F}_T$-measurable random variable $Z(T) > 0$ a.s. P. In that case we also write

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = Z(T) \quad \text{and} \quad \frac{\mathrm{d}P}{\mathrm{d}Q} = Z^{-1}(T) \quad \text{on } \mathcal{F}_T. \tag{1.4.1}$$

We first make a simple, but useful observation.

Lemma 1.26. *Suppose $Q \ll P$ with $\frac{\mathrm{d}Q}{\mathrm{d}P} = Z(T)$ on $\mathcal{F}_T$. Then*

$$Q \mid \mathcal{F}_t \ll P \mid \mathcal{F}_t \quad \textit{for all } t \in [0,T] \textit{ and}$$

$$Z(t) := \frac{\mathrm{d}(Q \mid \mathcal{F}_t)}{\mathrm{d}(P \mid \mathcal{F}_t)} = E_P[Z(T) \mid \mathcal{F}_t], \quad 0 \leq t \leq T. \tag{1.4.2}$$

In particular, $Z(t)$ is a P-martingale.

Proof of Theorem 1.24. Since $P(G) = 0 \Rightarrow Q(G) = 0$ for all $G \in \mathcal{F}_T \supseteq \mathcal{F}_t$, it is clear that $Q \mid \mathcal{F}_t \ll P \mid \mathcal{F}_t$. Choose $F \in \mathcal{F}_t$. Then

$$\begin{aligned} E_P\left[F \cdot E[Z(T) \mid \mathcal{F}_t]\right] &= E_P\left[E_P[FZ(T) \mid \mathcal{F}_t]\right] \\ &= E_P[FZ(T)] = E_Q[F] = E_P[F \cdot Z(t)]. \end{aligned}$$

Since this holds for all $F \in \mathcal{F}_t$ we conclude that

$$E_P[Z(T) \mid \mathcal{F}_t] = Z(t), \quad \text{as claimed.}$$

□

Lemma 1.27. *Suppose $Q \ll P$ with $\frac{\mathrm{d}Q}{\mathrm{d}P} = Z(T) > 0$ on $\mathcal{F}_T$. Let $X(t)$ be an adapted process such that $Z(t)X(t)$ is a martingale with respect to P. Then $X(t)$ is a martingale with respect to Q. Similarly, if $Z(t)X(t)$ is a* local *martingale with respect to P, then $X(t)$ is a* local *martingale with respect to Q.*

Proof of Theorem 1.24. We prove the last statement. Let $\tau \geq t$ be a stopping time, $\tau \leq T$. Then

$$\begin{aligned} E_Q[X(\tau) \mid \mathcal{F}_t] &= \frac{E[Z(T)X(\tau) \mid \mathcal{F}_t]}{E[Z(T) \mid \mathcal{F}_t]} = \frac{E[Z(\tau)X(\tau) \mid \mathcal{F}_t]}{Z(t)} = \frac{Z(t)X(t)}{Z(t)} \\ &= X(t) \quad \text{for all } t \in [0,T]. \end{aligned}$$

□