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An Introduction to Manifolds

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An Introduction to Manifolds

Second Edition

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Dedicated to the memory of Raoul Bott

Preface to the Second Edition

This is a completely revised edition, with more than fifty pages of new material scattered throughout. In keeping with the conventional meaning of chapters and sections, I have reorganized the book into twenty-nine sections in seven chapters. The main additions are Section 20 on the Lie derivative and interior multiplication, two intrinsic operations on a manifold too important to leave out, new criteria in Section 21 for the boundary orientation, and a new appendix on quaternions and the symplectic group.

Apart from correcting errors and misprints, I have thought through every proof again, clarified many passages, and added new examples, exercises, hints, and solutions. In the process, every section has been rewritten, sometimes quite drastically. The revisions are so extensive that it is not possible to enumerate them all here. Each chapter now comes with an introductory essay giving an overview of what is to come. To provide a timeline for the development of ideas, I have indicated whenever possible the historical origin of the concepts, and have augmented the bibliography with historical references.

Every author needs an audience. In preparing the second edition, I was particularly fortunate to have a loyal and devoted audience of two, George F. Leger and Jeffrey D. Carlson, who accompanied me every step of the way. Section by section, they combed through the revision and gave me detailed comments, corrections, and suggestions. In fact, the two hundred pages of feedback that Jeff wrote was in itself a masterpiece of criticism. Whatever clarity this book finally achieves results in a large measure from their effort. To both George and Jeff, I extend my sincere gratitude. I have also benefited from the comments and feedback of many other readers, including those of the copyeditor, David Kramer. Finally, it is a pleasure to thank Philippe Courrège, Mauricio Gutierrez, and Pierre Vogel for helpful discussions, and the Institut de Mathématiques de Jussieu and the Université Paris Diderot for hosting me during the revision. As always, I welcome readers' feedback.

Paris, France — *Loring W. Tu*
June 2010

Preface to the First Edition

It has been more than two decades since Raoul Bott and I published *Differential Forms in Algebraic Topology*. While this book has enjoyed a certain success, it does assume some familiarity with manifolds and so is not so readily accessible to the average first-year graduate student in mathematics. It has been my goal for quite some time to bridge this gap by writing an elementary introduction to manifolds assuming only one semester of abstract algebra and a year of real analysis. Moreover, given the tremendous interaction in the last twenty years between geometry and topology on the one hand and physics on the other, my intended audience includes not only budding mathematicians and advanced undergraduates, but also physicists who want a solid foundation in geometry and topology.

With so many excellent books on manifolds on the market, any author who undertakes to write another owes to the public, if not to himself, a good rationale. First and foremost is my desire to write a readable but rigorous introduction that gets the reader quickly up to speed, to the point where for example he or she can compute de Rham cohomology of simple spaces.

A second consideration stems from the self-imposed absence of point-set topology in the prerequisites. Most books laboring under the same constraint define a manifold as a subset of a Euclidean space. This has the disadvantage of making quotient manifolds such as projective spaces difficult to understand. My solution is to make the first four sections of the book independent of point-set topology and to place the necessary point-set topology in an appendix. While reading the first four sections, the student should at the same time study Appendix A to acquire the point-set topology that will be assumed starting in Section 5.

The book is meant to be read and studied by a novice. It is not meant to be encyclopedic. Therefore, I discuss only the irreducible minimum of manifold theory that I think every mathematician should know. I hope that the modesty of the scope allows the central ideas to emerge more clearly.

In order not to interrupt the flow of the exposition, certain proofs of a more routine or computational nature are left as exercises. Other exercises are scattered throughout the exposition, in their natural context. In addition to the exercises embedded in the text, there are problems at the end of each section. Hints and solutions to selected exercises and problems are gathered at the end of the book. I have starred the problems for which complete solutions are provided.

This book has been conceived as the first volume of a tetralogy on geometry and topology. The second volume is *Differential Forms in Algebraic Topology* cited above. I hope that Volume 3, *Differential Geometry: Connections, Curvature, and Characteristic Classes*, will soon see the light of day. Volume 4, *Elements of Equivariant Cohomology*, a long-running joint project with Raoul Bott before his passing away in 2005, is still under revision.

This project has been ten years in gestation. During this time I have benefited from the support and hospitality of many institutions in addition to my own; more specifically, I thank the French Ministère de l'Enseignement Supérieur et de la Recherche for a senior fellowship (bourse de haut niveau), the Institut Henri Poincaré, the Institut de Mathématiques de Jussieu, and the Departments of Mathematics at the École Normale Supérieure (rue d'Ulm), the Université Paris 7, and the Université de Lille, for stays of various length. All of them have contributed in some essential way to the finished product.

I owe a debt of gratitude to my colleagues Fulton Gonzalez, Zbigniew Nitecki, and Montserrat Teixidor i Bigas, who tested the manuscript and provided many useful comments and corrections, to my students Cristian Gonzalez-Martinez, Christopher Watson, and especially Aaron W. Brown and Jeffrey D. Carlson for their detailed errata and suggestions for improvement, to Ann Kostant of Springer and her team John Spiegelman and Elizabeth Loew for editing advice, typesetting, and manufacturing, respectively, and to Steve Schnably and Paul Gérardin for years of unwavering moral support. I thank Aaron W. Brown also for preparing the List of Notations and the TEX files for many of the solutions. Special thanks go to George Leger for his devotion to all of my book projects and for his careful reading of many versions of the manuscripts. His encouragement, feedback, and suggestions have been invaluable to me in this book as well as in several others. Finally, I want to mention Raoul Bott, whose courses on geometry and topology helped to shape my mathematical thinking and whose exemplary life is an inspiration to us all.

Medford, Massachusetts *Loring W. Tu*
June 2007

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A Brief Introduction

Undergraduate calculus progresses from differentiation and integration of functions on the real line to functions in the plane and in 3-space. Then one encounters vector-valued functions and learns about integrals on curves and surfaces. Real analysis extends differential and integral calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$. This book is about the extension of calculus from curves and surfaces to higher dimensions.

The higher-dimensional analogues of smooth curves and surfaces are called *manifolds*. The constructions and theorems of vector calculus become simpler in the more general setting of manifolds; gradient, curl, and divergence are all special cases of the exterior derivative, and the fundamental theorem for line integrals, Green's theorem, Stokes's theorem, and the divergence theorem are different manifestations of a single general Stokes's theorem for manifolds.

Higher-dimensional manifolds arise even if one is interested only in the three-dimensional space that we inhabit. For example, if we call a rotation followed by a translation an affine motion, then the set of all affine motions in $\mathbb{R}^3$ is a six-dimensional manifold. Moreover, this six-dimensional manifold is not $\mathbb{R}^6$.

We consider two manifolds to be topologically the same if there is a homeomorphism between them, that is, a bijection that is continuous in both directions. A topological invariant of a manifold is a property such as compactness that remains unchanged under a homeomorphism. Another example is the number of connected components of a manifold. Interestingly, we can use differential and integral calculus on manifolds to study the topology of manifolds. We obtain a more refined invariant called the de Rham cohomology of the manifold.

Our plan is as follows. First, we recast calculus on $\mathbb{R}^n$ in a way suitable for generalization to manifolds. We do this by giving meaning to the symbols dx, dy, and dz, so that they assume a life of their own, as *differential forms*, instead of being mere notations as in undergraduate calculus.

While it is not logically necessary to develop differential forms on $\mathbb{R}^n$ before the theory of manifolds—after all, the theory of differential forms on a manifold in Chapter 5 subsumes that on $\mathbb{R}^n$, from a pedagogical point of view it is advantageous to treat $\mathbb{R}^n$ separately first, since it is on $\mathbb{R}^n$ that the essential simplicity of differential forms and exterior differentiation becomes most apparent.

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Another reason that we do not delve into manifolds right away is so that in a course setting the students without a background in point-set topology can read Appendix A on their own while studying the calculus of differential forms on $\mathbb{R}^n$.

Armed with the rudiments of point-set topology, we define a manifold and derive various conditions for a set to be a manifold. A central idea of calculus is the approximation of a nonlinear object by a linear object. With this in mind, we investigate the relation between a manifold and its tangent spaces. Key examples are Lie groups and their Lie algebras.

Finally, we do calculus on manifolds, exploiting the interplay of analysis and topology to show on the one hand how the theorems of vector calculus generalize, and on the other hand, how the results on manifolds define new C^∞ invariants of a manifold, the de Rham cohomology groups.

The de Rham cohomology groups are in fact not merely C^∞ invariants, but also topological invariants, a consequence of the celebrated de Rham theorem that establishes an isomorphism between de Rham cohomology and singular cohomology with real coefficients. To prove this theorem would take us too far afield. Interested readers may find a proof in the sequel [4] to this book.

Chapter 1

Euclidean Spaces

The Euclidean space $\mathbb{R}^n$ is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. A good understanding of $\mathbb{R}^n$ is essential in generalizing differential and integral calculus to a manifold.

Euclidean space is special in having a set of standard global coordinates. This is both a blessing and a handicap. It is a blessing because all constructions on $\mathbb{R}^n$ can be defined in terms of the standard coordinates and all computations carried out explicitly. It is a handicap because, defined in terms of coordinates, it is often not obvious which concepts are intrinsic, i.e., independent of coordinates. Since a manifold in general does not have standard coordinates, only coordinate-independent concepts will make sense on a manifold. For example, it turns out that on a manifold of dimension n, it is not possible to integrate functions, because the integral of a function depends on a set of coordinates. The objects that can be integrated are differential forms. It is only because the existence of global coordinates permits an identification of functions with differential n-forms on $\mathbb{R}^n$ that integration of functions becomes possible on $\mathbb{R}^n$.

Our goal in this chapter is to recast calculus on $\mathbb{R}^n$ in a coordinate-free way suitable for generalization to manifolds. To this end, we view a tangent vector not as an arrow or as a column of numbers, but as a derivation on functions. This is followed by an exposition of Hermann Grassmann's formalism of alternating multilinear functions on a vector space, which lays the foundation for the theory of differential forms. Finally, we introduce differential forms on $\mathbb{R}^n$, together with two of their basic operations, the wedge product and the exterior derivative, and show how they generalize and simplify vector calculus in $\mathbb{R}^3$.

§1 Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds. For this reason, we begin with a review of C^∞ functions on $\mathbb{R}^n$.

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1.1 C^∞ Versus Analytic Functions

Write the coordinates on $\mathbb{R}^n$ as $x^1, \dots, x^n$ and let $p = (p^1, \dots, p^n)$ be a point in an open set U in $\mathbb{R}^n$. In keeping with the conventions of differential geometry, the indices on coordinates are *superscripts*, not subscripts. An explanation of the rules for superscripts and subscripts is given in Subsection 4.7.

Definition 1.1. Let k be a nonnegative integer. A real-valued function $f\colon U \to \mathbb{R}$ is said to be C^k *at* $p \in U$ if its partial derivatives

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$$

of all orders $j \le k$ exist and are continuous at p. The function $f\colon U \to \mathbb{R}$ is C^∞ *at* p if it is C^k for all $k \ge 0$; in other words, its partial derivatives $\partial^j f / \partial x^{i_1} \cdots \partial x^{i_j}$ of all orders exist and are continuous at p. A vector-valued function $f\colon U \to \mathbb{R}^m$ is said to be C^k *at* p if all of its component functions $f^1, \dots, f^m$ are C^k at p. We say that $f\colon U \to \mathbb{R}^m$ is C^k *on* U if it is C^k at every point in U. A similar definition holds for a C^∞ function on an open set U. We treat the terms "C^∞" and "smooth" as synonymous.

Example 1.2.
(i) A C^0 function on U is a continuous function on U.
(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0, \\ \text{undefined} & \text{for } x = 0. \end{cases}$$

Thus the function f is C^0 but not C^1 at $x = 0$.
(iii) Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \int_0^x f(t)\,dt = \int_0^x t^{1/3}\,dt = \frac{3}{4}x^{4/3}.$$

Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.
(iv) The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

A *neighborhood* of a point in $\mathbb{R}^n$ is an open set containing the point. The function f is *real-analytic* at p if in some neighborhood of p it is equal to its Taylor series at p:

$$f(x) = f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j)$$
$$+ \cdots + \frac{1}{k!}\sum_{i_1,\dots,i_k} \frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}(p)(x^{i_1} - p^{i_1}) \cdots (x^{i_k} - p^{i_k}) + \cdots,$$

in which the general term is summed over all $1 \leq i_1, \ldots, i_k \leq n$.

A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots,$$

then term-by-term differentiation gives

$$f'(x) = \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots.$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0.

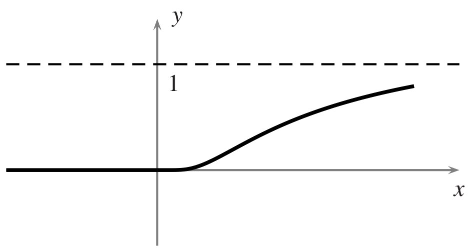

Fig. 1.1. A C^∞ function all of whose derivatives vanish at 0.

Example 1.3 (*A C^∞ function very flat at* 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

(See Figure 1.1.) By induction, one can show that f is C^∞ on $\mathbb{R}$ and that the derivatives $f^{(k)}(0)$ are equal to 0 for all $k \geq 0$ (Problem 1.2).

The Taylor series of this function at the origin is identically zero in any neighborhood of the origin, since all derivatives $f^{(k)}(0)$ equal 0. Therefore, $f(x)$ cannot be equal to its Taylor series and $f(x)$ is not real-analytic at 0.

1.2 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. In the lemma below, we prove the very first case, in which the Taylor series consists of only the constant term $f(p)$.

We say that a subset S of $\mathbb{R}^n$ is *star-shaped* with respect to a point p in S if for every x in S, the line segment from p to x lies in S (Figure 1.2).