

Universitext

UTX

Loring W. Tu

An Introduction to Manifolds

Second Edition

 Springer

Universitext

Editorial Board
(North America):

S. Axler
K.A. Ribet

For other titles in this series, go to
www.springer.com/series/223

Loring W. Tu

An Introduction to Manifolds

Second Edition

 Springer

Loring W. Tu
Department of Mathematics
Tufts University
Medford, MA 02155
loring.tu@tufts.edu

Editorial board:

Sheldon Axler, San Francisco State University
Vincenzo Capasso, Università degli Studi di Milano
Carles Casacuberta, Universitat de Barcelona
Angus MacIntyre, Queen Mary, University of London
Kenneth Ribet, University of California, Berkeley
Claude Sabbah, CNRS, École Polytechnique
Endre Süli, University of Oxford
Wojbor Woyczyński, Case Western Reserve University

ISBN 978-1-4419-7399-3 e-ISBN 978-1-4419-7400-6
DOI 10.1007/978-1-4419-7400-6
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010936466

Mathematics Subject Classification (2010): 58-01, 58Axx, 58A05, 58A10, 58A12

© Springer Science+ Business Media, LLC 2011

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+ Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Dedicated to the memory of Raoul Bott

Preface to the Second Edition

This is a completely revised edition, with more than fifty pages of new material scattered throughout. In keeping with the conventional meaning of chapters and sections, I have reorganized the book into twenty-nine sections in seven chapters. The main additions are Section 20 on the Lie derivative and interior multiplication, two intrinsic operations on a manifold too important to leave out, new criteria in Section 21 for the boundary orientation, and a new appendix on quaternions and the symplectic group.

Apart from correcting errors and misprints, I have thought through every proof again, clarified many passages, and added new examples, exercises, hints, and solutions. In the process, every section has been rewritten, sometimes quite drastically. The revisions are so extensive that it is not possible to enumerate them all here. Each chapter now comes with an introductory essay giving an overview of what is to come. To provide a timeline for the development of ideas, I have indicated whenever possible the historical origin of the concepts, and have augmented the bibliography with historical references.

Every author needs an audience. In preparing the second edition, I was particularly fortunate to have a loyal and devoted audience of two, George F. Leger and Jeffrey D. Carlson, who accompanied me every step of the way. Section by section, they combed through the revision and gave me detailed comments, corrections, and suggestions. In fact, the two hundred pages of feedback that Jeff wrote was in itself a masterpiece of criticism. Whatever clarity this book finally achieves results in a large measure from their effort. To both George and Jeff, I extend my sincere gratitude. I have also benefited from the comments and feedback of many other readers, including those of the copyeditor, David Kramer. Finally, it is a pleasure to thank Philippe Courrège, Mauricio Gutierrez, and Pierre Vogel for helpful discussions, and the Institut de Mathématiques de Jussieu and the Université Paris Diderot for hosting me during the revision. As always, I welcome readers' feedback.

Paris, France
June 2010

Loring W. Tu

Preface to the First Edition

It has been more than two decades since Raoul Bott and I published *Differential Forms in Algebraic Topology*. While this book has enjoyed a certain success, it does assume some familiarity with manifolds and so is not so readily accessible to the average first-year graduate student in mathematics. It has been my goal for quite some time to bridge this gap by writing an elementary introduction to manifolds assuming only one semester of abstract algebra and a year of real analysis. Moreover, given the tremendous interaction in the last twenty years between geometry and topology on the one hand and physics on the other, my intended audience includes not only budding mathematicians and advanced undergraduates, but also physicists who want a solid foundation in geometry and topology.

With so many excellent books on manifolds on the market, any author who undertakes to write another owes to the public, if not to himself, a good rationale. First and foremost is my desire to write a readable but rigorous introduction that gets the reader quickly up to speed, to the point where for example he or she can compute de Rham cohomology of simple spaces.

A second consideration stems from the self-imposed absence of point-set topology in the prerequisites. Most books laboring under the same constraint define a manifold as a subset of a Euclidean space. This has the disadvantage of making quotient manifolds such as projective spaces difficult to understand. My solution is to make the first four sections of the book independent of point-set topology and to place the necessary point-set topology in an appendix. While reading the first four sections, the student should at the same time study Appendix A to acquire the point-set topology that will be assumed starting in Section 5.

The book is meant to be read and studied by a novice. It is not meant to be encyclopedic. Therefore, I discuss only the irreducible minimum of manifold theory that I think every mathematician should know. I hope that the modesty of the scope allows the central ideas to emerge more clearly.

In order not to interrupt the flow of the exposition, certain proofs of a more routine or computational nature are left as exercises. Other exercises are scattered throughout the exposition, in their natural context. In addition to the exercises embedded in the text, there are problems at the end of each section. Hints and solutions

to selected exercises and problems are gathered at the end of the book. I have starred the problems for which complete solutions are provided.

This book has been conceived as the first volume of a tetralogy on geometry and topology. The second volume is *Differential Forms in Algebraic Topology* cited above. I hope that Volume 3, *Differential Geometry: Connections, Curvature, and Characteristic Classes*, will soon see the light of day. Volume 4, *Elements of Equivariant Cohomology*, a long-running joint project with Raoul Bott before his passing away in 2005, is still under revision.

This project has been ten years in gestation. During this time I have benefited from the support and hospitality of many institutions in addition to my own; more specifically, I thank the French Ministère de l'Enseignement Supérieur et de la Recherche for a senior fellowship (bourse de haut niveau), the Institut Henri Poincaré, the Institut de Mathématiques de Jussieu, and the Departments of Mathematics at the École Normale Supérieure (rue d'Ulm), the Université Paris 7, and the Université de Lille, for stays of various length. All of them have contributed in some essential way to the finished product.

I owe a debt of gratitude to my colleagues Fulton Gonzalez, Zbigniew Nitecki, and Montserrat Teixidor i Bigas, who tested the manuscript and provided many useful comments and corrections, to my students Cristian Gonzalez-Martinez, Christopher Watson, and especially Aaron W. Brown and Jeffrey D. Carlson for their detailed errata and suggestions for improvement, to Ann Kostant of Springer and her team John Spiegelman and Elizabeth Loew for editing advice, typesetting, and manufacturing, respectively, and to Steve Schnably and Paul Gérardin for years of unwavering moral support. I thank Aaron W. Brown also for preparing the List of Notations and the \TeX files for many of the solutions. Special thanks go to George Leger for his devotion to all of my book projects and for his careful reading of many versions of the manuscripts. His encouragement, feedback, and suggestions have been invaluable to me in this book as well as in several others. Finally, I want to mention Raoul Bott, whose courses on geometry and topology helped to shape my mathematical thinking and whose exemplary life is an inspiration to us all.

Contents

Preface to the Second Edition	vii
Preface to the First Edition	ix
A Brief Introduction	1
<hr/>	
Chapter 1 Euclidean Spaces	
<hr/>	
§1 Smooth Functions on a Euclidean Space	3
1.1 C^∞ Versus Analytic Functions	4
1.2 Taylor's Theorem with Remainder	5
Problems	8
§2 Tangent Vectors in \mathbb{R}^n as Derivations	10
2.1 The Directional Derivative	10
2.2 Germs of Functions	11
2.3 Derivations at a Point	13
2.4 Vector Fields	14
2.5 Vector Fields as Derivations	16
Problems	17
§3 The Exterior Algebra of Multivectors	18
3.1 Dual Space	19
3.2 Permutations	20
3.3 Multilinear Functions	22
3.4 The Permutation Action on Multilinear Functions	23
3.5 The Symmetrizing and Alternating Operators	24
3.6 The Tensor Product	25
3.7 The Wedge Product	26
3.8 Anticommutativity of the Wedge Product	27
3.9 Associativity of the Wedge Product	28
3.10 A Basis for k -Covectors	31

Problems	32
§4 Differential Forms on \mathbb{R}^n	34
4.1 Differential 1-Forms and the Differential of a Function	34
4.2 Differential k -Forms	36
4.3 Differential Forms as Multilinear Functions on Vector Fields	37
4.4 The Exterior Derivative	38
4.5 Closed Forms and Exact Forms	40
4.6 Applications to Vector Calculus	41
4.7 Convention on Subscripts and Superscripts	44
Problems	44

Chapter 2 Manifolds

§5 Manifolds	48
5.1 Topological Manifolds	48
5.2 Compatible Charts	49
5.3 Smooth Manifolds	52
5.4 Examples of Smooth Manifolds	53
Problems	57
§6 Smooth Maps on a Manifold	59
6.1 Smooth Functions on a Manifold	59
6.2 Smooth Maps Between Manifolds	61
6.3 Diffeomorphisms	63
6.4 Smoothness in Terms of Components	63
6.5 Examples of Smooth Maps	65
6.6 Partial Derivatives	67
6.7 The Inverse Function Theorem	68
Problems	70
§7 Quotients	71
7.1 The Quotient Topology	71
7.2 Continuity of a Map on a Quotient	72
7.3 Identification of a Subset to a Point	73
7.4 A Necessary Condition for a Hausdorff Quotient	73
7.5 Open Equivalence Relations	74
7.6 Real Projective Space	76
7.7 The Standard C^∞ Atlas on a Real Projective Space	79
Problems	81

Chapter 3 The Tangent Space

§8 The Tangent Space	86
-----------------------------------	----

8.1	The Tangent Space at a Point	86
8.2	The Differential of a Map	87
8.3	The Chain Rule	88
8.4	Bases for the Tangent Space at a Point	89
8.5	A Local Expression for the Differential	91
8.6	Curves in a Manifold	92
8.7	Computing the Differential Using Curves	95
8.8	Immersions and Submersions	96
8.9	Rank, and Critical and Regular Points	96
	Problems	98
§9	Submanifolds	100
9.1	Submanifolds	100
9.2	Level Sets of a Function	103
9.3	The Regular Level Set Theorem	105
9.4	Examples of Regular Submanifolds	106
	Problems	108
§10	Categories and Functors	110
10.1	Categories	110
10.2	Functors	111
10.3	The Dual Functor and the Multicovector Functor	113
	Problems	114
§11	The Rank of a Smooth Map	115
11.1	Constant Rank Theorem	115
11.2	The Immersion and Submersion Theorems	118
11.3	Images of Smooth Maps	120
11.4	Smooth Maps into a Submanifold	124
11.5	The Tangent Plane to a Surface in \mathbb{R}^3	125
	Problems	127
§12	The Tangent Bundle	129
12.1	The Topology of the Tangent Bundle	129
12.2	The Manifold Structure on the Tangent Bundle	132
12.3	Vector Bundles	133
12.4	Smooth Sections	136
12.5	Smooth Frames	137
	Problems	139
§13	Bump Functions and Partitions of Unity	140
13.1	C^∞ Bump Functions	140
13.2	Partitions of Unity	145
13.3	Existence of a Partition of Unity	146
	Problems	147
§14	Vector Fields	149
14.1	Smoothness of a Vector Field	149

14.2	Integral Curves	152
14.3	Local Flows	154
14.4	The Lie Bracket	157
14.5	The Pushforward of Vector Fields	159
14.6	Related Vector Fields	159
	Problems	161

Chapter 4 Lie Groups and Lie Algebras

§15	Lie Groups	164
15.1	Examples of Lie Groups	164
15.2	Lie Subgroups	167
15.3	The Matrix Exponential	169
15.4	The Trace of a Matrix	171
15.5	The Differential of \det at the Identity	174
	Problems	174
§16	Lie Algebras	178
16.1	Tangent Space at the Identity of a Lie Group	178
16.2	Left-Invariant Vector Fields on a Lie Group	180
16.3	The Lie Algebra of a Lie Group	182
16.4	The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$	183
16.5	The Pushforward of Left-Invariant Vector Fields	184
16.6	The Differential as a Lie Algebra Homomorphism	185
	Problems	187

Chapter 5 Differential Forms

§17	Differential 1-Forms	190
17.1	The Differential of a Function	191
17.2	Local Expression for a Differential 1-Form	191
17.3	The Cotangent Bundle	192
17.4	Characterization of C^∞ 1-Forms	193
17.5	Pullback of 1-Forms	195
17.6	Restriction of 1-Forms to an Immersed Submanifold	197
	Problems	199
§18	Differential k-Forms	200
18.1	Differential Forms	200
18.2	Local Expression for a k -Form	202
18.3	The Bundle Point of View	203
18.4	Smooth k -Forms	203
18.5	Pullback of k -Forms	204
18.6	The Wedge Product	205
18.7	Differential Forms on a Circle	206

18.8	Invariant Forms on a Lie Group	207
	Problems	208
§19	The Exterior Derivative	210
19.1	Exterior Derivative on a Coordinate Chart	211
19.2	Local Operators	211
19.3	Existence of an Exterior Derivative on a Manifold	212
19.4	Uniqueness of the Exterior Derivative	213
19.5	Exterior Differentiation Under a Pullback	214
19.6	Restriction of k -Forms to a Submanifold	216
19.7	A Nowhere-Vanishing 1-Form on the Circle	216
	Problems	218
§20	The Lie Derivative and Interior Multiplication	221
20.1	Families of Vector Fields and Differential Forms	221
20.2	The Lie Derivative of a Vector Field	223
20.3	The Lie Derivative of a Differential Form	226
20.4	Interior Multiplication	227
20.5	Properties of the Lie Derivative	229
20.6	Global Formulas for the Lie and Exterior Derivatives	232
	Problems	233

Chapter 6 Integration

§21	Orientations	236
21.1	Orientations of a Vector Space	236
21.2	Orientations and n -Covectors	238
21.3	Orientations on a Manifold	240
21.4	Orientations and Differential Forms	242
21.5	Orientations and Atlases	245
	Problems	246
§22	Manifolds with Boundary	248
22.1	Smooth Invariance of Domain in \mathbb{R}^n	248
22.2	Manifolds with Boundary	250
22.3	The Boundary of a Manifold with Boundary	253
22.4	Tangent Vectors, Differential Forms, and Orientations	253
22.5	Outward-Pointing Vector Fields	254
22.6	Boundary Orientation	255
	Problems	256
§23	Integration on Manifolds	260
23.1	The Riemann Integral of a Function on \mathbb{R}^n	260
23.2	Integrability Conditions	262
23.3	The Integral of an n -Form on \mathbb{R}^n	263
23.4	Integral of a Differential Form over a Manifold	265

23.5	Stokes's Theorem	269
23.6	Line Integrals and Green's Theorem	271
	Problems	272

Chapter 7 De Rham Theory

§24	De Rham Cohomology	274
24.1	De Rham Cohomology	274
24.2	Examples of de Rham Cohomology	276
24.3	Diffeomorphism Invariance	278
24.4	The Ring Structure on de Rham Cohomology	279
	Problems	280
§25	The Long Exact Sequence in Cohomology	281
25.1	Exact Sequences	281
25.2	Cohomology of Cochain Complexes	283
25.3	The Connecting Homomorphism	284
25.4	The Zig-Zag Lemma	285
	Problems	287
§26	The Mayer–Vietoris Sequence	288
26.1	The Mayer–Vietoris Sequence	288
26.2	The Cohomology of the Circle	292
26.3	The Euler Characteristic	295
	Problems	295
§27	Homotopy Invariance	296
27.1	Smooth Homotopy	296
27.2	Homotopy Type	297
27.3	Deformation Retractions	299
27.4	The Homotopy Axiom for de Rham Cohomology	300
	Problems	301
§28	Computation of de Rham Cohomology	302
28.1	Cohomology Vector Space of a Torus	302
28.2	The Cohomology Ring of a Torus	303
28.3	The Cohomology of a Surface of Genus g	306
	Problems	310
§29	Proof of Homotopy Invariance	311
29.1	Reduction to Two Sections	311
29.2	Cochain Homotopies	312
29.3	Differential Forms on $M \times \mathbb{R}$	312
29.4	A Cochain Homotopy Between i_0^* and i_1^*	314
29.5	Verification of Cochain Homotopy	315
	Problems	316

Appendices

§A	Point-Set Topology	317
	A.1 Topological Spaces	317
	A.2 Subspace Topology	320
	A.3 Bases	321
	A.4 First and Second Countability	323
	A.5 Separation Axioms	324
	A.6 Product Topology	326
	A.7 Continuity	327
	A.8 Compactness	329
	A.9 Boundedness in \mathbb{R}^n	332
	A.10 Connectedness	332
	A.11 Connected Components	333
	A.12 Closure	334
	A.13 Convergence	336
	Problems	337
§B	The Inverse Function Theorem on \mathbb{R}^n and Related Results	339
	B.1 The Inverse Function Theorem	339
	B.2 The Implicit Function Theorem	339
	B.3 Constant Rank Theorem	343
	Problems	344
§C	Existence of a Partition of Unity in General	346
§D	Linear Algebra	349
	D.1 Quotient Vector Spaces	349
	D.2 Linear Transformations	350
	D.3 Direct Product and Direct Sum	351
	Problems	352
§E	Quaternions and the Symplectic Group	353
	E.1 Representation of Linear Maps by Matrices	354
	E.2 Quaternionic Conjugation	355
	E.3 Quaternionic Inner Product	356
	E.4 Representations of Quaternions by Complex Numbers	356
	E.5 Quaternionic Inner Product in Terms of Complex Components	357
	E.6 \mathbb{H} -Linearity in Terms of Complex Numbers	357
	E.7 Symplectic Group	358
	Problems	359
	Solutions to Selected Exercises Within the Text	361
	Hints and Solutions to Selected End-of-Section Problems	367

List of Notations	387
References	395
Index	397

A Brief Introduction

Undergraduate calculus progresses from differentiation and integration of functions on the real line to functions in the plane and in 3-space. Then one encounters vector-valued functions and learns about integrals on curves and surfaces. Real analysis extends differential and integral calculus from \mathbb{R}^3 to \mathbb{R}^n . This book is about the extension of calculus from curves and surfaces to higher dimensions.

The higher-dimensional analogues of smooth curves and surfaces are called *manifolds*. The constructions and theorems of vector calculus become simpler in the more general setting of manifolds; gradient, curl, and divergence are all special cases of the exterior derivative, and the fundamental theorem for line integrals, Green's theorem, Stokes's theorem, and the divergence theorem are different manifestations of a single general Stokes's theorem for manifolds.

Higher-dimensional manifolds arise even if one is interested only in the three-dimensional space that we inhabit. For example, if we call a rotation followed by a translation an affine motion, then the set of all affine motions in \mathbb{R}^3 is a six-dimensional manifold. Moreover, this six-dimensional manifold is not \mathbb{R}^6 .

We consider two manifolds to be topologically the same if there is a homeomorphism between them, that is, a bijection that is continuous in both directions. A topological invariant of a manifold is a property such as compactness that remains unchanged under a homeomorphism. Another example is the number of connected components of a manifold. Interestingly, we can use differential and integral calculus on manifolds to study the topology of manifolds. We obtain a more refined invariant called the de Rham cohomology of the manifold.

Our plan is as follows. First, we recast calculus on \mathbb{R}^n in a way suitable for generalization to manifolds. We do this by giving meaning to the symbols dx , dy , and dz , so that they assume a life of their own, as *differential forms*, instead of being mere notations as in undergraduate calculus.

While it is not logically necessary to develop differential forms on \mathbb{R}^n before the theory of manifolds—after all, the theory of differential forms on a manifold in Chapter 5 subsumes that on \mathbb{R}^n , from a pedagogical point of view it is advantageous to treat \mathbb{R}^n separately first, since it is on \mathbb{R}^n that the essential simplicity of differential forms and exterior differentiation becomes most apparent.

Another reason that we do not delve into manifolds right away is so that in a course setting the students without a background in point-set topology can read Appendix A on their own while studying the calculus of differential forms on \mathbb{R}^n .

Armed with the rudiments of point-set topology, we define a manifold and derive various conditions for a set to be a manifold. A central idea of calculus is the approximation of a nonlinear object by a linear object. With this in mind, we investigate the relation between a manifold and its tangent spaces. Key examples are Lie groups and their Lie algebras.

Finally, we do calculus on manifolds, exploiting the interplay of analysis and topology to show on the one hand how the theorems of vector calculus generalize, and on the other hand, how the results on manifolds define new C^∞ invariants of a manifold, the de Rham cohomology groups.

The de Rham cohomology groups are in fact not merely C^∞ invariants, but also topological invariants, a consequence of the celebrated de Rham theorem that establishes an isomorphism between de Rham cohomology and singular cohomology with real coefficients. To prove this theorem would take us too far afield. Interested readers may find a proof in the sequel [4] to this book.

Chapter 1

Euclidean Spaces

The Euclidean space \mathbb{R}^n is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like \mathbb{R}^n . A good understanding of \mathbb{R}^n is essential in generalizing differential and integral calculus to a manifold.

Euclidean space is special in having a set of standard global coordinates. This is both a blessing and a handicap. It is a blessing because all constructions on \mathbb{R}^n can be defined in terms of the standard coordinates and all computations carried out explicitly. It is a handicap because, defined in terms of coordinates, it is often not obvious which concepts are intrinsic, i.e., independent of coordinates. Since a manifold in general does not have standard coordinates, only coordinate-independent concepts will make sense on a manifold. For example, it turns out that on a manifold of dimension n , it is not possible to integrate functions, because the integral of a function depends on a set of coordinates. The objects that can be integrated are differential forms. It is only because the existence of global coordinates permits an identification of functions with differential n -forms on \mathbb{R}^n that integration of functions becomes possible on \mathbb{R}^n .

Our goal in this chapter is to recast calculus on \mathbb{R}^n in a coordinate-free way suitable for generalization to manifolds. To this end, we view a tangent vector not as an arrow or as a column of numbers, but as a derivation on functions. This is followed by an exposition of Hermann Grassmann's formalism of alternating multilinear functions on a vector space, which lays the foundation for the theory of differential forms. Finally, we introduce differential forms on \mathbb{R}^n , together with two of their basic operations, the wedge product and the exterior derivative, and show how they generalize and simplify vector calculus in \mathbb{R}^3 .

§1 Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds. For this reason, we begin with a review of C^∞ functions on \mathbb{R}^n .

1.1 C^∞ Versus Analytic Functions

Write the coordinates on \mathbb{R}^n as x^1, \dots, x^n and let $p = (p^1, \dots, p^n)$ be a point in an open set U in \mathbb{R}^n . In keeping with the conventions of differential geometry, the indices on coordinates are *superscripts*, not subscripts. An explanation of the rules for superscripts and subscripts is given in Subsection 4.7.

Definition 1.1. Let k be a nonnegative integer. A real-valued function $f: U \rightarrow \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\frac{\partial^j f}{\partial x^{i_1} \dots \partial x^{i_j}}$$

of all orders $j \leq k$ exist and are continuous at p . The function $f: U \rightarrow \mathbb{R}$ is C^∞ at p if it is C^k for all $k \geq 0$; in other words, its partial derivatives $\partial^j f / \partial x^{i_1} \dots \partial x^{i_j}$ of all orders exist and are continuous at p . A vector-valued function $f: U \rightarrow \mathbb{R}^m$ is said to be C^k at p if all of its component functions f^1, \dots, f^m are C^k at p . We say that $f: U \rightarrow \mathbb{R}^m$ is C^k on U if it is C^k at every point in U . A similar definition holds for a C^∞ function on an open set U . We treat the terms “ C^∞ ” and “smooth” as synonymous.

Example 1.2.

- (i) A C^0 function on U is a continuous function on U .
 (ii) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0, \\ \text{undefined} & \text{for } x = 0. \end{cases}$$

Thus the function f is C^0 but not C^1 at $x = 0$.

- (iii) Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$g(x) = \int_0^x f(t) dt = \int_0^x t^{1/3} dt = \frac{3}{4}x^{4/3}.$$

Then $g'(x) = f(x) = x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x = 0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.

- (iv) The polynomial, sine, cosine, and exponential functions on the real line are all C^∞ .

A *neighborhood* of a point in \mathbb{R}^n is an open set containing the point. The function f is *real-analytic* at p if in some neighborhood of p it is equal to its Taylor series at p :

$$\begin{aligned} f(x) = & f(p) + \sum_i \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) \\ & + \dots + \frac{1}{k!} \sum_{i_1, \dots, i_k} \frac{\partial^k f}{\partial x^{i_1} \dots \partial x^{i_k}}(p)(x^{i_1} - p^{i_1}) \dots (x^{i_k} - p^{i_k}) + \dots, \end{aligned}$$

in which the general term is summed over all $1 \leq i_1, \dots, i_k \leq n$.

A real-analytic function is necessarily C^∞ , because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x) = \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots,$$

then term-by-term differentiation gives

$$f'(x) = \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots.$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on \mathbb{R} whose graph, though not horizontal, is “very flat” near 0 in the sense that all of its derivatives vanish at 0.

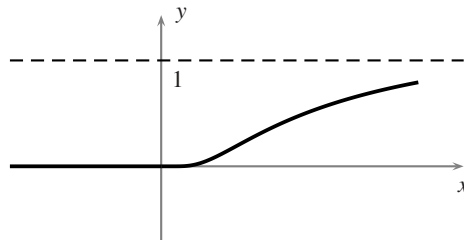


Fig. 1.1. A C^∞ function all of whose derivatives vanish at 0.

Example 1.3 (A C^∞ function very flat at 0). Define $f(x)$ on \mathbb{R} by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

(See Figure 1.1.) By induction, one can show that f is C^∞ on \mathbb{R} and that the derivatives $f^{(k)}(0)$ are equal to 0 for all $k \geq 0$ (Problem 1.2).

The Taylor series of this function at the origin is identically zero in any neighborhood of the origin, since all derivatives $f^{(k)}(0)$ equal 0. Therefore, $f(x)$ cannot be equal to its Taylor series and $f(x)$ is not real-analytic at 0.

1.2 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. In the lemma below, we prove the very first case, in which the Taylor series consists of only the constant term $f(p)$.

We say that a subset S of \mathbb{R}^n is *star-shaped* with respect to a point p in S if for every x in S , the line segment from p to x lies in S (Figure 1.2).

- [click Lonely Planet Scotland pdf, azw \(kindle\), epub](#)
 - [Belling the Cat: Essays, Reports and Opinions online](#)
 - [The Sea Watch online](#)
 - [read Applied Cryptography: Protocols, Algorithms, and Source Code in C \(2nd Edition\) pdf, azw \(kindle\), epub](#)
 - [Sharks: A Picture Book of Fun Facts about Sharks for free](#)
 - [read online Antike](#)
-
- <http://paulczajak.com/?library/Lonely-Planet-Scotland.pdf>
 - <http://berttrotman.com/library/Belling-the-Cat--Essays--Reports-and-Opinions.pdf>
 - <http://korplast.gr/lib/The-Sea-Watch.pdf>
 - <http://cambridgebrass.com/?freebooks/Applied-Cryptography--Protocols--Algorithms--and-Source-Code-in-C--2nd-Edition-.pdf>
 - <http://bestarthritiscare.com/library/Sharks--A-Picture-Book-of-Fun-Facts-about-Sharks.pdf>
 - <http://academialanguagebar.com/?ebooks/Antike.pdf>