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An Introduction to Compactness Results in Symplectic Field Theory

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Preface

This text originates from special topics lectures I gave at Michigan State University on Symplectic Field Theory during the Fall of 2005 and the Spring of 2006 for graduate students after their third year. The first lecture covered compactness results, while the second was about polyfold theory.

Symplectic Field Theory yields powerful invariants for symplectic and contact manifolds. It is constructed using suitable moduli spaces of pseudoholomorphic curves, and it generalizes Floer Homology, Gromov–Witten theory and Contact Homology. The first paper on Symplectic Field Theory (SFT) was the 113-page survey by Yakov Eliashberg, Alexandre Givental and Helmut Hofer in the year 2000 [20]. As of now, a decade later, the general theory of SFT is still in development.

The concept of a polyfold was introduced by H. Hofer, K. Wysocki and E. Zehnder to address the numerous technical challenges in SFT in a systematic way. The reader is referred to the articles [39–42]. As a first application of polyfold theory H. Hofer, K. Wysocki and E. Zehnder recently gave a complete construction of Gromov–Witten theory in full generality [43]. Pseudoholomorphic curves are solutions to a nonlinear version of the Cauchy Riemann equations. Before a solution space of a nonlinear system of elliptic partial differential equations can be equipped with the structure of a polyfold it is necessary to understand its compactness properties. In the case of pseudoholomorphic curves this is the subject of this lecture.

Pseudoholomorphic curves have become a useful tool in symplectic geometry, and they were introduced by M. Gromov in his ground breaking 1985 paper [30]. Gromov's work is based on understanding moduli spaces of pseudoholomorphic curves on compact Riemann surfaces in a compact symplectic manifold. The subject of this text is to construct and describe the compactification relevant for SFT by F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder [12], a generalization of Gromov's result. Finally, the theory of polyfolds provides a general analytic framework for certain spaces which admit no smooth manifold structure, such as compactifications of spaces of pseudoholomorphic curves.

Andreas Floer was the first to recognize the importance of pseudoholomorphic curves on noncompact Riemann surfaces in his celebrated work on the Arnold conjecture [27] in 1988. Since then numerous different flavors of Floer Homology have been studied, and SFT is a construction in the same spirit.

Another important step consists of the work of Helmut Hofer in 1993 [33] on the Weinstein conjecture in dimension three. The Weinstein conjecture states that the Reeb vector field on any closed contact manifold has a periodic orbit. The main tool in H. Hofer's paper are pseudoholomorphic curves on the complex plane into the symplectization of a contact manifold M. In dimension three the Weinstein conjecture was proved by Clifford Taubes [67, 68] in 2007/2009 using Seiberg Witten equations. Interestingly, the gauge theoretic and the pseudoholomorphic curve stories are closely related. This is apparent from the proof that Seiberg–Witten Floer Homology and M. Hutching's Embedded Contact Homology (in some sense a version of SFT) are isomorphic (see [52, 69–73] for more information on ECH).

Hofer showed that the existence of a nonconstant pseudoholomorphic plane with finite energy implies the existence of a periodic orbit of the Reeb vector field, and he proved such existence results under some additional assumptions on M (see also [3] for other developments).

In the last decade special cases of the general Symplectic Field Theory construction and different flavors of it have been established and studied, already with far reaching applications. See [17, 18, 21–24, 44, 45, 52, 53] for a sample of the already large number of works on the subject.

In this text we will give a proof of the compactness results in SFT as in the paper [12], but with considerably more details and background material. We also present a version for curves with boundary (see [16, 26] for related results). The SFT compactness result describes what a sequence of pseudoholomorphic curves converges to (in a suitable sense), and it provides a description of the compactified moduli space. The outcome of this compactification procedure is the space of all holomorphic buildings which we discuss in detail. We also present all the necessary background material from hyperbolic geometry of surfaces. The purpose is to give a unified and detailed presentation which is currently not available in the literature. Hopefully this text makes the beginnings of Symplectic Field Theory more accessible.

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Contents

Chapter 1
Riemann Surfaces

The purpose of this chapter is to provide the material necessary for understanding convergence of Riemann surfaces in the sense of Deligne–Mumford. If we want to discuss the convergence behavior of a sequence of J-holomorphic curves (u_k) we need to take into account that their domains are all different Riemann surfaces (S, j_k). The aim is to establish a suitable notion of convergence for these as well. The discussion here follows Thurston's approach [74] utilizing hyperbolic geometry similar to C. Hummel's proof of Gromov's compactness theorem [46] (see also [5, 13] and [75] as general references). The details, however, are spread out over the literature so we give a thorough and unified presentation.

1.1 Smooth and Noded Riemann Surfaces

Definition 1.1 An atlas on a smooth surface S with charts $\varphi_\alpha : U_\alpha \to \mathbb{C}$, $\bigcup_\alpha U_\alpha = S$ is called *conformal* if the transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic. A *conformal structure* on S is a maximal conformal atlas. A smooth surface S together with a conformal structure is called a *Riemann surface*. A continuous map $f : S \to S'$ between two Riemann surfaces is called *holomorphic* if in local coordinates $\{U_\alpha, \varphi_\alpha\}$ on S and $\{U'_\beta, \varphi'_\beta\}$ on S' the maps $\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}$ are holomorphic whenever they are defined. A holomorphic map $f : S \to S'$ is called *conformal* if its derivative is never zero.

It is common practice to identify a coordinate patch $U_\alpha \subset S$ with its image $\varphi_\alpha(U_\alpha) \subset \mathbb{C}$ suppressing the coordinate map φ_α in the notation. This makes sense if we consider local objects and notions which are invariant under conformal maps. For example, if $f : S \to \mathbb{R}$ is a smooth function on a Riemann surface then we may define 'f is subharmonic' by the requirement that $\frac{\partial^2 f}{\partial z \partial \bar{z}} \geq 0$. This is well-defined if

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for any locally defined conformal map $z = h(w)$ the map $f \circ h$ satisfies $\frac{\partial^2 (f \circ h)}{\partial w \partial \bar{w}} \geq 0$. But this follows from

$$\frac{\partial^2}{\partial w \partial \bar{w}}(f \circ h) = \frac{\partial^2 f}{\partial z \partial \bar{z}} \frac{\partial \bar{h}}{\partial \bar{w}} \frac{\partial h}{\partial w} = \frac{\partial^2 f}{\partial z \partial \bar{z}} \left| \frac{\partial h}{\partial w} \right|^2 \geq 0.$$

From this it is clear that the local expression '$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 1$' for example would not make sense globally on a Riemann surface. The following statements hold (the proofs are trivial and we leave them to the reader):

(1) Every Riemann surface is orientable.
(2) Assume S is a Riemann surface, S' is a smooth surface and $\pi : S' \to S$ is a local diffeomorphism. Then there exists a unique conformal structure on S' such that π becomes holomorphic.
(3) Application: Every covering space of a Riemann surface can be made into a Riemann surface in a canonical fashion.
(4) Let S' be a Riemann surface, let S be a smooth surface and let $\pi : S' \to S$ be a covering. If every covering transformation $\Psi : S' \to S'$ is holomorphic then there is a unique conformal structure on S such that π becomes holomorphic.

There are different ways to look at Riemann surfaces. We will elaborate on some of them and explain why they are equivalent to Definition 1.1.

Definition 1.2 (Almost complex structure-Alternative definition of Riemann surface) Let W be a differentiable manifold. An *almost complex structure* on M is a smooth section j of the vector bundle $\mathrm{Hom}(TM, TM) \to M$, $\mathrm{Hom}(TM, TM)_z = \mathcal{L}(T_z M, T_z M)$ such that $j^2(z) = -\mathrm{Id}_{T_z M}$ for all $z \in M$. A pair (S, j) consisting of a smooth surface S and an almost complex structure j is called a *Riemann surface*.

The two notions of Riemann surface are equivalent: Assume S is a smooth surface together with a conformal structure. Then we can define an almost complex structure j on S as follows: Let $z \in S$. Let $\varphi_\alpha : U_\alpha \to \mathbb{C}$ be a coordinate chart belonging to the conformal structure. We define

$$j(z) := \big(D\varphi_\alpha(z)\big)^{-1} \circ i \circ D\varphi_\alpha(z). \tag{1.1}$$

Show as an exercise that this definition does not depend on the choice of the coordinate chart φ_α. This procedure also works for general complex manifolds, i.e. smooth manifolds of even dimension which admit an atlas with holomorphic coordinate transition maps. The converse procedure would be constructing a conformal structure on a manifold with a given almost complex structure j such that (1.1) holds.

Definition 1.3 Assume that W is a manifold with an almost complex structure j. If W admits an atlas such that the transition maps between coordinate patches are all holomorphic, and if j is then given by (1.1) then it is called a *complex structure* or an *integrable almost complex structure*.

Remark 1.4 The existence question of an almost complex structure on a given even dimensional manifold M is rather a topological question while the existence question of an integrable almost complex structure is of analytical nature, and it is usually more difficult to answer. For example, it is known that the only spheres admitting almost complex structures are S^2 and S^6. By our remarks above every almost complex structure on S^2 is integrable. On the other hand, it is an open question whether S^6 admits any complex structure.

A celebrated theorem due to Newlander and Nirenberg [57] asserts that an almost complex structure j on a manifold W is integrable if and only if the Nijenhuis tensor, defined by

$$N(X,Y) := [X,Y] + j[jX,Y] + j[X,jY] - [jX,jY]$$

vanishes for all vector fields X, Y on W. Here $[\cdot,\cdot]$ denotes the Lie bracket. If W is 2-dimensional then the Nijenhuis tensor does vanish so that Definitions 1.1 and 1.2 are indeed equivalent. Another way of defining Riemann surfaces is taken by Lipman Bers in his lecture notes [10]. He defines a Riemann surface to be a smooth surface together with a distinguished family of functions which he calls analytic functions. His point of view is that the conformal structure determines which functions $f : S \to \mathbb{C}$ are holomorphic or not.

We will prove the integrability of almost complex structures on surfaces apart from two assertions where a complete proof would lead us too far astray and which we will address later in more detail. The higher dimensional case (the Newlander–Nirenberg theorem) is much more difficult. In order to get an idea, the reader may look at the introduction of [76]. The integrability result is the following:

Proposition 1.5 *Let $\mathcal{J}$ be the set of real 2×2 matrices j such that $j^2 = -\mathrm{Id}$ and $\{v, jv\}$ is a positively oriented basis of $\mathbb{R}^2$ whenever $0 \neq v \in \mathbb{R}^2$. Denote the standard complex structure on $\mathbb{R}^2 \approx \mathbb{C}$ by i and let $D := \{s+it \in \mathbb{C} \mid s^2+t^2 < 1\}$. Assume that $j : D \to \mathcal{J}$ is a smooth map. Then there is a diffeomorphism ψ between suitable open neighborhoods of $0 \in D$ with $\psi(0) = 0$, $\partial_s\psi(0) = 1$ and*

$$d\psi(s+it)\circ j(s+it) = i \circ d\psi(s+it). \tag{1.2}$$

Proof **Step 1:** *Reformulate (1.2) using the Beltrami equation.*

The almost complex structure j is of the form

$$j = \begin{pmatrix} b & -a \\ c & -b \end{pmatrix}$$

where $a, b, c : D \to \mathbb{R}$ are (smooth) functions satisfying $a, c > 0$, $ac - b^2 = 1$. Using a linear isomorphism we may assume that $j(0) = i$, i.e. $b(0) = 0$ and $a(0) = c(0) = 1$. Define a complex valued function μ by

$$\mu := \frac{c-a-2ib}{a+c+2}. \tag{1.3}$$

The differential equation (1.2) is equivalent to the so-called Beltrami equation

$$\bar{\partial}\psi = \mu\partial\psi, \tag{1.4}$$

where

$$\partial\zeta := \frac{1}{2}(\partial_s\zeta - i\partial_t\zeta), \qquad \bar{\partial}\zeta = \frac{1}{2}(\partial_s\zeta + i\partial_t\zeta).$$

Indeed, we compute

$$\begin{aligned} d\psi \circ j &= (b\partial_s\psi + c\partial_t\psi)\,ds - (a\partial_s\psi + b\partial_t\psi)\,dt \\ &= i\,d\psi \\ &= i\partial_s\psi\,ds + i\partial_t\psi\,dt. \end{aligned}$$

This is the same as

$$(b-i)\partial_s\psi + c\partial_t\psi = 0, \qquad a\partial_s\psi + (b+i)\partial_t\psi = 0,$$

and these two equations are equivalent in view of $ac - b^2 = 1$. If we expand the Beltrami equation

$$(a+c+2)\bar{\partial}\psi = (c-a-2ib)\partial\psi$$

we arrive directly at the two above equations. If a solution ψ to the Beltrami equation (or to $\psi_* j = i\psi_*$) satisfies in addition $\psi(0) = 0$ and $\partial_s\psi(0) = 1$ then it is a local diffeomorphism near 0 since $\partial_s\psi(0)$ and $\partial_t\psi(0)$ are linear independent over the real numbers.

Step 2: *Existence of a local solution to the Beltrami equation of class $C^{1,\alpha}$.*

Let $D \subset \mathbb{C}$ be the closed unit disk and let $0 < \alpha \le 1$. We have the following result:

Proposition 1.6 *Let $C := \{\eta \in C^{1,\alpha}(D,\mathbb{C}) \,|\, \eta(e^{i\theta}) \in \mathbb{R}e^{\frac{3}{2}i\theta}\}$. The operator*

$$\begin{aligned} \bar{\partial} : C &\longrightarrow C^{\alpha}(D,\mathbb{C}) \\ \eta &\longmapsto \partial_s\eta + i\partial_t\eta \end{aligned}$$

is a surjective Fredholm operator of index four.

The reader should consult Appendices A5 and A6 in [4] for a proof.

In fact, the kernel of the above operator consists of all polynomials of the form $\eta(z) = a_0 + a_1 z + \bar{a}_1 z^2 + \bar{a}_0 z^3$. Then the following operator is a Banach space isomorphism:

$$\begin{aligned} T : C &\longrightarrow C^{\alpha}(D,\mathbb{C}) \times \mathbb{C} \times \mathbb{C} \\ T\eta &:= \big(\bar{\partial}\eta, \eta(0), \partial_s\eta(0)\big). \end{aligned}$$

We define now the following smooth map:

$$\Phi : C^\alpha(D,\mathbb{C}) \times C^\alpha(D,\mathbb{C}) \longrightarrow C^\alpha(D,\mathbb{C}) \times C^\alpha(D,\mathbb{C})$$
$$\Phi(f,\mu) := \big(f - \mu\big(\partial\big(T^{-1}(f,0,1)\big)\big), \mu\big).$$

We are done with step 2 if we can find a solution (f,μ) to the equation

$$\Phi(f,\mu) = (0,\mu).$$

Indeed, assume there is such a solution. Then

$$f = \mu\big(\partial\big(T^{-1}(f,0,1)\big)\big)$$

and $\psi := T^{-1}(f,0,1)$ satisfies

$$\bar\partial\psi = f, \qquad \psi(0) = 0, \qquad \partial_s\psi(0) = 1$$

so that

$$\bar\partial\psi = \mu\partial\psi, \qquad \psi(0) = 0, \qquad \partial_s\psi(0) = 1.$$

We compute the derivative of Φ at $(0,0)$:

$$D\Phi(0,0)(h,\lambda) = \big(h - \lambda\big(\partial\big(T^{-1}(0,0,1)\big)\big), \lambda\big).$$

Write $\tau = T^{-1}(0,0,1) = z + z^2$ and recall the above characterization of the kernel of the Cauchy Riemann operator on the space C. We must have $a_0 = 0$ in view of $\tau(0) = 0$, hence $\tau = a_1 z + \bar a_1 z^2$. Then $1 = \partial_s\tau(0) = \partial\tau(0) = a_1$ leads to $\tau(z) = z + z^2$ and $\partial\tau = 1 + 2z$. Hence

$$\big(D\Phi(0,0)(h,\lambda)\big)(z) = \big(h(z) - \lambda(z)(1+2z), \lambda(z)\big)$$

which has a bounded inverse

$$\big(D\Phi(0,0)^{-1}(g,\kappa)\big)(z) = \big(g(z) + \kappa(z)(1+2z), \kappa(z)\big).$$

By the inverse function theorem, the map Φ is a diffeomorphism between suitable open neighborhoods U_α, V_α of $(0,0) \in C^\alpha(D,\mathbb{C}) \times C^\alpha(D,\mathbb{C})$. Let $\beta : D \to [0,1]$ be a smooth function such that $\beta(z) \equiv 1$ for $|z| \le 1/4$ and $\beta(z) \equiv 0$ if $|z| \ge 1/2$. If $\varepsilon \in (0,1]$ we define

$$\mu_\varepsilon(z) := \mu(z)\beta\left(\frac{z}{\varepsilon}\right),$$

where μ is the function as in (1.3). Recall that we have arranged earlier for $\mu(0) = 0$ so that for $|z| \le \varepsilon/2$

$$\big|\mu(z)\big| = \left|\int_0^1 \mu'(\tau z) z\, d\tau\right| \le c|z|, \qquad c = \sup_{B_{\varepsilon/2}(0)} |\mu'|.$$

We obtain

$$\begin{aligned}\left|\mu_\varepsilon'(z)\right| &\le \left|\mu'(z)\beta\left(\varepsilon^{-1}z\right)\right| + \varepsilon^{-1}\left|\mu(z)\beta'\left(\varepsilon^{-1}z\right)\right| \\ &\le c + c\left\|\beta'\right\|_{C^0}\end{aligned}$$

which amounts to a C^1-bound on the functions μ_ε independent of ε. Then $\|\mu_\varepsilon\|_{C^\alpha(D)} \to 0$ as $\varepsilon \to 0$ (show this as an exercise!). Then $(0, \mu_\varepsilon) \in V_\alpha$ if ε is sufficiently small. Let $(f, \nu) := \Phi^{-1}(0, \mu_\varepsilon)$. Then $\nu = \mu_\varepsilon$, and $\psi := T^{-1}(f, 0, 1)$ solves the desired equation on a sufficiently small disk centered at the origin.

Step 3 (*Sketch only*): *Regularity, i.e. if μ smooth then so is the solution to the Beltrami equation.*

Let ψ be the function of class $C^{1,\alpha}$ constructed in step 2. Because its derivative is nonsingular in the origin it is a local C^1-diffeomorphism between suitable neighborhoods $U, V \subset \mathbb{C}$ of 0. Let φ be its inverse, so that

$$d\varphi(z) + j\big(\varphi(z)\big) \circ d\varphi(z) \circ i = 0,$$

in other words, φ is a j-holomorphic curve. By regularity of j-holomorphic curves, if φ and j are of class $W^{k,p}$ then u is actually of class $W^{k+1,p}$. In particular, if j is smooth then so is φ and also ψ. We will not address the topic of regularity of pseudoholomorphic curves. The interested reader may consult Appendix A4 in [4] or [50]. □

The Beltrami equation is closely related to the following result:

Proposition 1.7 (Existence of isothermal parameters on a surface) *Let S be an oriented surface with orientation form σ and a (smooth) Riemannian metric g. Then for any point $p \in S$ there is a local coordinate chart $\phi : U(p) \to \mathbb{R}^2$ such that $\psi^*(ds \wedge dt) = f\sigma$ for some positive function f (i.e. ψ is orientation preserving) and $\psi^*(ds^2 + dt^2) = hg$, where h is also a positive functions.*

Proof If we write out the metric g in arbitrary local coordinates as $E\,ds^2 + 2F\,ds\,dt + G\,dt^2$ with $E, G > 0$ and $EG - F^2 > 0$ then we can define a (local) almost complex structure by

$$j_g = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix}$$

(which corresponds to rotation by 90° in the tangent planes). This is globally defined if the underlying surface is oriented. If we choose now local coordinates so that j_g is transformed to the standard structure i then in these coordinates the metric g transforms to one which differs from the Euclidean metric by multiplication with a positive function (we say 'conformal to the Euclidean metric'). □

We summarize our previous discussion:

A conformal structure on a surface induces an almost complex structure in a natural way via (1.1). On the other hand, if S is a surface endowed with an almost complex structure j then there is a conformal structure on S which induces j.

If g is a Riemannian metric on an oriented surface S then there is a canonical almost complex structure j_g on S, which just rotates vectors in each tangent plane by 90°. Using the result about isothermal parameters we can find a conformal structure on S, and the metric g is conformal to the Euclidean metric in these coordinates. The almost complex structure induced by the conformal structure coincides with j_g. If g, h are two Riemannian metrics on S then the induced almost complex structures j_g and j_h are equal if and only if g and h are conformal, i.e. if there is a positive function λ such that $h = \lambda g$.

A priori there is no distinguished metric g which induces a given complex structure j. We will later define the concept of a stable surface (which includes most surfaces) where complex structures are in 1:1 correspondence with complete Riemannian metrics with constant sectional curvature -1. We will call such a metric **the** *Poincaré metric* of the corresponding Riemann surface. The advantage of this viewpoint is that the class of these metrics on a given surface S can be understood geometrically while this is not so clear for the set of all complex structures.

Proposition 1.7 goes back to Carl Friedrich Gauss (1820s) if the metric g is real analytic (see [65] for Gauss' argument), to Korn and Lichtenstein (1914, 1916) for metrics of class $C^{1,\alpha}$ and Morrey, Bers, Ahlfors (1938, [8] 1960) for g merely of class L^∞ in which case the coordinate change is of class $W^{1,p}$ for some $p > 2$. For a classical proof of the Korn–Lichtenstein result see [15] and [65]. The Beltrami equation is reformulated as an integral equation. After proving some estimates a solution operator to the Beltrami equation can then be constructed. The proof in the L^∞ case is in the same spirit and uses the usual techniques for proving L^p estimates (see [8]). Our proof was taken from [50] because it is more in tune with the topics covered in this lecture. After having looked at Riemann surfaces from different angles, let us close this section with a brief overview of the topics ahead of us.

We are interested in the Riemann moduli space $\mathcal{M}_g$ of equivalence classes of closed Riemann surfaces of genus $g \geq 2$ and other related moduli spaces (surfaces with boundary, with points removed etc.). We want to regard two Riemann surfaces (S, j) and (S', j') as equivalent if there is a diffeomorphism $\phi : S \to S'$ such that $\phi_* j = j' \phi_*$. Let $\mathcal{J}(S)$ be the set of all complex structures on a fixed smooth closed surface S of genus $g \geq 2$. The group of all diffeomorphisms $\mathrm{Diff}(S)$ of S acts on $\mathcal{J}(S)$ via $j \mapsto (\phi^{-1})_* \circ j \circ \phi_*$. Then $\mathcal{M}_g$ can be identified with

$$\mathcal{J}(S)/\mathrm{Diff}(S).$$

Dividing by the connected component of the identity map $\mathrm{Diff}_0(S) \subset \mathrm{Diff}(S)$ yields the so-called Teichmüller space. We will define a slightly larger space $\overline{\mathcal{M}_g}$ which will serve as a compactification of the space $\mathcal{M}_g$. The additional objects in $\overline{\mathcal{M}_g}$ are the so-called noded Riemann surfaces. We will define a suitable notion of convergence ('Deligne–Mumford convergence') such that every sequence in $\overline{\mathcal{M}_g}$ has a convergent subsequence. This is the statement of the Deligne–Mumford compactness result. The space $\mathcal{M}_g$ can actually be equipped with a metric such that DM-convergence is the same as convergence with respect to this metric, and the completion of $\mathcal{M}_g$ is $\overline{\mathcal{M}_g}$, but we will not pursue this topic at the moment. We also point out that we are not dealing with any kind of issues regarding smooth structure on the compactified Riemann space. In fact, the space $\overline{\mathcal{M}_g}$ carries an orbifold structure for $g \geq 2$ (see [62]). We will largely follow the notation of the paper [12]. The proof of the Deligne–Mumford compactness result will follow the presentation in [46].

Definition 1.8 Let (S, j) be a Riemann surface. A *marking* of the surface S is a finite sequence of points $(x_1, \ldots, x_n)$, $x_j \in S$. We use the notation $M = (x_1, \ldots, x_n)$. If a marking $(x_1, \ldots, x_n)$ is given then we call $\mathbf{S} = (S, j, M)$ a *Riemann surface with marked points*.

Note that the ordering of the points $x_1, \ldots, x_n$ is part of the structure (unless we say otherwise). We now define the objects which need to be added in order to compactify the Riemann moduli space.

Consider a (possibly disconnected) Riemann surface $\mathbf{S} = (S, j, M, D)$ with marked points $M \cup D$ so that $M \cap D = \emptyset$. The points in the set D are called *special marked points* or *nodal points* because they have the following additional features: The set D consists of an even number of points which are organized in pairs

$$D = \{\overline{d}_1, \underline{d}_1, \ldots, \overline{d}_k, \underline{d}_k\}.$$

As for the marked points in M the ordering of the points is part of the structure, i.e. merely exchanging the order of two marked points in M yields a Riemann surface which we consider different from the original one. The above notation suggests that in the set D we are not interested in the particular ordering of the pairs $(\overline{d}_1, \underline{d}_1), \ldots, (\overline{d}_k, \underline{d}_k)$. Moreover we will also identify $(\overline{d}_j, \underline{d}_j)$ with $(\underline{d}_j, \overline{d}_j)$. This ordering convention is the one from the article [12]. A priori other choices are possible.

Definition 1.9 We call a Riemann surface (S, j, M, D) with marked points $M = (x_1, \ldots, x_n)$ and special marked points $D = \{\overline{d}_1, \underline{d}_1, \ldots, \overline{d}_k, \underline{d}_k\}$ a *noded Riemann surface*. Two noded Riemann surfaces $\mathbf{S} = (S, j, M, D)$ and $\mathbf{S}' = (S', j', M', D')$ with $M = (x_1, \ldots, x_n)$ and $M' = (x'_1, \ldots, x'_n)$ are called (*conformally*) *equivalent* if there is a diffeomorphism $\varphi : S \to S'$ such that $\varphi_* j = j'\varphi_*$, $\varphi(x_j) = x'_j$ for all $1 \leq j \leq n$ and $\varphi(D) = D'$ mapping pairs to pairs.

We use the notation $\mathbf{S} = [S, j, M, D]$ if we refer to the equivalence class of a noded Riemann surface (S, j, M, D). Sometimes we will also call $[S, j, M, D]$ a noded Riemann surface.

We can associate to a noded Riemann surface a singular surface as follows:

Definition 1.10 Let $\mathbf{S}$ be a noded Riemann surface. We define

$$\hat{S}_D := S/\{\overline{d}_j \sim \underline{d}_j \mid 1 \leq j \leq k\}$$

and we call $\hat{S}_D$ the *singular surface associated to* $\mathbf{S}$. We call the noded surface $\mathbf{S}$ *connected* if its singular surface is. If $\mathbf{S}$ is connected we define its *arithmetic*

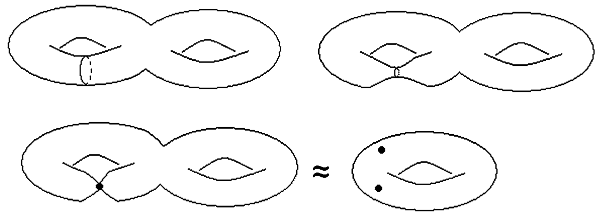


Fig. 1.1 Illustrating convergence of Riemann surfaces to a noded surface

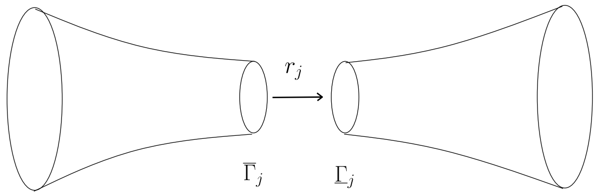


Fig. 1.2 A pair of decorated nodal points

genus g by

$$g := \frac{\#D}{2} - C + \sum_{j=1}^{C} g_j + 1,$$

where C is the number of connected components of the surface S and $g_1, \ldots, g_C$ are the genera of the connected components of S. If $\mathbf{S}$ is a noded Riemann surface with arithmetic genus g and $\mu := \#M$ then the pair (g, μ) is called the *signature* of $\mathbf{S}$.

Example 1.11 If S is closed and connected, and if $D = \emptyset$ then the arithmetic genus coincides with the genus of S. If S is the 2-torus with a pair of nodal points then $g = 2$. The meaning of the arithmetic genus comes from the notion of convergence in the sense of Deligne and Mumford. It allows in a sequence of Riemann surfaces for closed curves to shrink to a point, creating a noded Riemann surface (see Fig. 1.1).

We will define an additional structure for noded Riemann surfaces, the notion of a decorated node.

Definition 1.12 (Decorated noded Riemann surface) Let $\mathbf{S} = (S, j, M, D)$ be a noded Riemann surface. We say $\mathbf{S}$ is *decorated* if every pair $\{\underline{d}_j, \overline{d}_j\}$ in D carries the following additional structure: A map

$$r_j : \overline{\Gamma}_j := \left(T_{\overline{d}_j} S \backslash \{0\}\right)/\mathbb{R}_+^* \longrightarrow \underline{\Gamma}_j := \left(T_{\underline{d}_j} S \backslash \{0\}\right)/\mathbb{R}_+^*, \quad \mathbb{R}_+^* := (0, +\infty)$$

satisfying $r_j(e^{i\theta} z) = e^{-i\theta} r_j(z)$ for all $z \in \overline{\Gamma}_j$ (see Fig. 1.2).

An equivalence between decorated noded Riemann surfaces has to preserve the decoration maps r_j. We introduce the following notation for spaces of Riemann

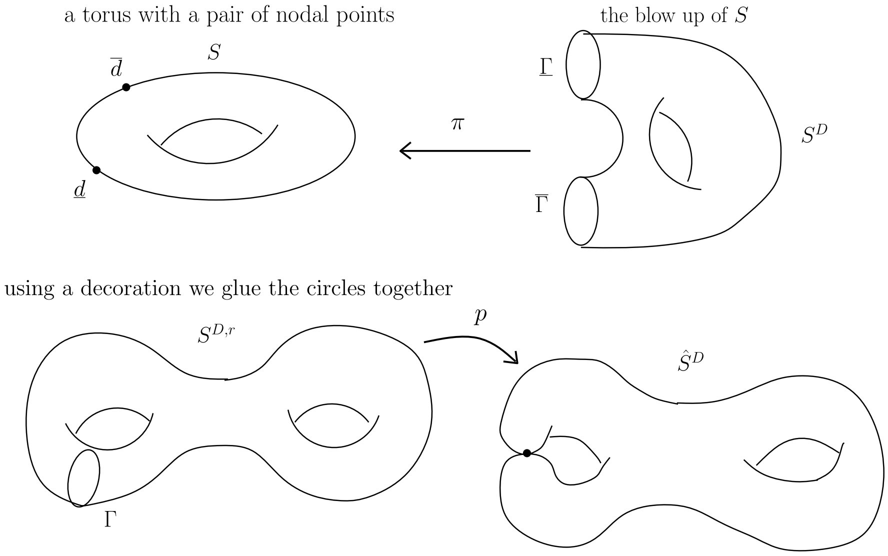


Fig. 1.3 Blow up of a surface and singular surface

surfaces (S, j, M, D) of signature (g, μ):

$$\left[\begin{array}{ll} \mathcal{M}_{g,\mu} & \text{smooth Riemann surfaces, i.e. } D = \emptyset \\ \overline{\mathcal{M}}_{g,\mu} & \text{noded Riemann surfaces} \\ \overline{\mathcal{M}}^{\$}_{g,\mu} & \text{noded decorated Riemann surfaces.} \end{array}\right. \tag{1.5}$$

Usually, we will also require that the surfaces are stable which means that for each connected component twice the genus plus the number of marked points is greater than two (more on this later). Assume $\mathbf{S} = (S, j, M, D)$ is a noded Riemann surface. We define a new surface S^D, called the *blow-up* of $\mathbf{S}$ as follows: We remove all the points $\{\underline{d}_j, \overline{d}_j\}$ from S, then we compactify the resulting surface by adding the circles $\overline{\Gamma}_j$, $\underline{\Gamma}_j$ defined above. Then there is the canonical projection $\pi : S^D \to S$ which collapses the boundary circles $\overline{\Gamma}_j$, $\underline{\Gamma}_j$ to the corresponding points $\overline{d}_j$, $\underline{d}_j$. The projection π induces a conformal structure on the interior of S^D which, however, degenerates along the boundary circles $\overline{\Gamma}_j$, $\underline{\Gamma}_j$.

If $\mathbf{S}$ comes with a decoration (in which case we write $(\mathbf{S}, r)$) we can glue the circles $\overline{\Gamma}_j$, $\underline{\Gamma}_j$ together with the decoration maps r_j, and we obtain a closed surface $S^{D,r}$. The genus of $S^{D,r}$ then equals the arithmetic genus of $\mathbf{S}$, and we obtain a canonical projection $p : S^{D,r} \to \hat{S}^D$ collapsing the circle $\Gamma_j = \{\overline{\Gamma}_j, \underline{\Gamma}_j\}$ to the double point $d_j = \{\overline{d}_j, \underline{d}_j\}$. The projection p also induces a conformal structure on $S^{D,r}$ with the special circles Γ_j removed. The surface $S^{D,r}$ is called the *deformation* of the noded decorated surface $(\mathbf{S}, r)$.

The reason why we are introducing noded surfaces is that they are needed to compactify the moduli space of smooth Riemann surfaces. The main result of this chapter is the following:

Theorem *Assume* $\mathbf{S}_n = (S_n, j_n, M_n)$ *is a sequence of smooth marked stable Riemann surfaces of signature* (g, μ). *Then* $(\mathbf{S}_n)_{n\in\mathbb{N}}$ *has a subsequence which converges to a stable decorated noded Riemann surface* $\mathbf{S} = (S, j, M, D, r)$ *of signature* (g, μ).

At this point we have not introduced the notion of convergence of a sequence of Riemann surfaces yet, but we will do so later in this chapter. For stable surfaces, complex structures correspond to complete metrics with sectional curvature -1 (Poincaré metrics). We will present a proof of this fact in the next two sections of this chapter. We will also show that every such surface is isometric to a simple model (pair of pants decomposition, Bers' theorem). A sequence of smooth surfaces will converge to a non-smooth noded surface if the Poincaré metrics degenerate in the limit along a finite union of closed curves. As we will show, after allowing reparametrizations of the surfaces, this is the only phenomenon preventing convergence of the metrics. This requires a detailed study of the degeneration process which in turn requires quite a bit of (elementary) hyperbolic geometry.

Theorem 1.13 (Uniformization theorem for simply connected Riemann surfaces) *Let* (S, j) *be a simply connected Riemann surface without boundary. Then* (S, j) *is conformally equivalent to exactly one of the following Riemann surfaces*:

(1) *The complex plane* $\mathbb{C}$,
(2) *The upper half plane* $H^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$,
(3) *The Riemann sphere* $S^2 = \mathbb{C} \cup \{\infty\}$.

By conformal equivalence we mean for example in case (1) the existence of a diffeomorphism $\psi : S \to \mathbb{C}$ such that $\psi_* j = i\psi_*$. The proof of the theorem would take too much time for this lecture, good references are [6, 10].

From the above theorem one can derive the following result (see [6]):

Theorem 1.14 (Uniformization theorem) *If* S *is a connected Riemann surface without boundary then* S *is conformally equivalent to either*

(1) $\mathbb{C}$,
(2) $\mathbb{C}\backslash\{0\}$,
(3) $\mathbb{C}/L$, *where* L *is a lattice*,
(4) $\mathbb{C}P^1 = S^2 = \mathbb{C} \cup \{\infty\}$,
(5) H^+/G, *where* $H^+ = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ *and* G *is a group of conformal maps of* H^+ *acting freely and properly discontinuously*.

Proof (Assuming the uniformization theorem for simply connected surfaces) Let S be a connected Riemann surface without boundary. Let $p : \hat{S} \to S$ be its universal covering, and let $G \approx \pi_1(S)$ be the group of covering transformations, i.e.

$$G = \{g : \hat{S} \to \hat{S} \mid g \text{ is conformal and } p \circ g = p\}$$

so that $S \approx \hat{S}/G$ ($\hat{S}$ carries the conformal structure induced by S which makes all covering transformations conformal, '$\approx$' means 'conformally equivalent'). The group G acts freely and properly discontinuously. Acting freely means that apart from the identity map, all $g \in G$ are fixed point free. Acting properly discontinuously means that every point x has a neighborhood U such that the set $\{g \in G \mid g(U) \cap U \neq \emptyset\}$ is finite. By Theorem 1.13 there are only three possibilities for $\hat{S}$. If $\hat{S}$ is conformally equivalent to the 2-sphere then we use the fact that the group of all conformal transformations of S^2 is given by

$$\mathrm{Conf}(S^2) = \left\{ z \overset{\phi}{\mapsto} \frac{az+b}{cz+d} \,\middle|\, a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\}.$$

Since all of them have fixed points we conclude that the group of covering transformations is trivial and S is conformally equivalent to the 2-sphere as well. If $\hat{S} \approx \mathbb{C}$ then G is a properly discontinuous group of parallel translations. There are three of them

(1) $G = \{\mathrm{Id}\}$ so that $S = \mathbb{C}$,
(2) G is the infinite cyclic group generated by $z \mapsto z + b$ for some $b \neq 0$ so that S is an infinite cylinder conformally equivalent to the punctured plane,
(3) G is the abelian group generated by two translations $z \mapsto z + b_1$ and $z \mapsto z + b_2$ where $b_1/b_2 \notin \mathbb{R}$ so that S is a torus.

In all the other cases, S is a quotient of the hyperbolic plane. □

The following result is a special case of the so-called Hopf–Killing theorem, the Riemannian geometry version of the uniformization theorem (see [28], Chap. 3.F for the proof):

Theorem 1.15 *Let S be a simply connected surface without boundary with a complete Riemannian metric of constant sectional curvature $K = -1$. Then S is isometric to $H^+ := \{z = x + iy \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the metric $g_{H^+} = y^{-2}(dx^2 + dy^2)$.*

In the case $K = 0$ the surface S is isometric to $\mathbb{C}$ with the Euclidean metric, and in the case $K = +1$ it is the 2-sphere with the usual metric. We will not need these cases here, nor do we need the corresponding statements in dimensions greater than two. The following proposition follows from the proof of Theorem 1.15 in [28], and it is a local version of the theorem:

Proposition 1.16 *Let S be a surface of sectional curvature -1. If $\partial S \neq \emptyset$ then assume that all components of the boundary are closed geodesics. Then every point in $S \backslash \partial S$ has an open neighborhood which is isometric to an open subset of H^+. Every point $q \in \partial S$ has an open neighborhood which is isometric to a set of the form $U \cap \{z \in H^+ \mid \mathrm{Re}(z) \geq 0\}$ where $U \subset H^+$ is some open neighborhood of i.*

1.2 Riemann Surfaces and Hyperbolic Geometry

1.2.1 Stable Surfaces

Let S be a smooth oriented surface. In this section, S may have several connected components or $\partial S \neq \emptyset$ as well.

Definition 1.17 A *finite extension* of S is a smooth orientation preserving embedding $i : S \hookrightarrow \Sigma$ into a compact oriented surface Σ such that $\Sigma \setminus i(S)$ is a finite set. The elements in the set $\Sigma \setminus i(S)$ are called *punctures* of S. If $S_1, \ldots, S_n$ are the connected components of S, and if $\Sigma_1, \ldots, \Sigma_n$ are the corresponding components of a finite extension Σ then let g_j be the genus of Σ_j, let m_j be the number of boundary components of Σ_j, and let n_j be the number of points in $\Sigma_j \setminus i(S_j)$. The list $\{(g_j, m_j, n_j)\}_{1 \leq j \leq n}$ is called the *signature* of S. We say that S is *stable* if for each $1 \leq j \leq n$ we have

$$n_j > \chi(\Sigma_j) = 2 - 2g_j - m_j$$

(i.e. at least three puncture points if Σ_j is a 2-sphere, at least two on a disk, at least one on an annulus or a torus) or

$$2g_j + m_j + n_j \geq 3.$$

Remark 1.18 Noded and marked Riemann surfaces fit into this picture if we interpret the marked and nodal points as punctures. Then Σ would be the original surface and S would be $\Sigma \setminus (M \cup D)$. A noded Riemann surface $\mathbf{S} = (S, j, M, D)$ is stable if and only if the automorphism group of each connected component is finite, i.e.

$$\#\left\{\phi \in \mathrm{Diff}(S_j) \mid \phi_* j = j \phi_*,\ \phi(x) = x\ \forall x \in M_j,\ \phi(D_j) = D_j\right\} < \infty.$$

For example in the case of the two sphere, all biholomorphic maps $\phi : S^2 \to S^2$, $S^2 = \mathbb{C} \cup \{\infty\}$ are the Möbius transformations

$$\phi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

In this case a direct calculation shows that the automorphism group becomes finite if we demand that a set of at least three points is left invariant. If we consider three marked points $M = (0, 1, \infty)$ then the automorphism group consists of the identity map only because the set of marked points is ordered and each automorphism ϕ has to satisfy $\phi(1) = 1$, $\phi(0) = 0$ and $\phi(\infty) = \infty$. If we declare ∞ a marked point and $\{0, 1\}$ nodal points then the automorphism group has exactly two elements, namely the identity map and $\phi(z) = 1 - z$. As for compact Riemann surfaces of genus $g \geq 2$ much more is known: The number of all orientation preserving automorphisms cannot exceed $84(g - 1)$. This result is due to Adolf Hurwitz (1888), and it is known as Hurwitz's automorphism theorem [47].

Definition 1.19 (Hyperbolic metric) A *hyperbolic metric* on a smooth oriented surface S is a complete Riemann metric of constant sectional curvature -1 such that each boundary component is a closed geodesic. Such a surface together with a hyperbolic metric is also called a *hyperbolic surface*. A *finite extension of a complex structure* j on S is a finite extension $i : S \hookrightarrow \Sigma$ such that $i_* j$ extends to a complex structure on Σ. We say j is of *finite type* if it admits a finite extension.

We mention a few facts about complete Riemannian manifolds (see [54], Chap. 10 or [28]): First,we recall that a Riemannian manifold W is called (geodesically) complete if every geodesic can be extended to the whole real line. This is equivalent to the fact that for any point $p \in W$ the exponential map $\exp_p$ is defined on all of $T_p W$. If we define for $p, q \in W$

$$\varrho(p,q) := \inf_{\gamma} \left\{ \int_0^1 \left|\dot{\gamma}(t)\right| dt \,\middle|\, \gamma : [0,1] \to W \right.$$
$$\left. \text{is a piecewise smooth path with } \gamma(0) = p,\ \gamma(1) = q \right\}$$

then (W, ϱ) is a metric space, and the metric ϱ induces the usual topology on W. Geodesic completeness of W is equivalent to completeness of (W, ϱ) as a metric space.[1] If W is complete then the Hopf–Rinow theorem asserts that any two points in a connected component can be joined by a minimal geodesic. We remark that a minimal geodesic joining two points is in general not unique (take for example two antipodal points on the 2-sphere). Moreover, the property that any two points can be joined by a minimal geodesic does not imply completeness.

Theorem 1.20 (Existence and Uniqueness of a hyperbolic metric) *Let S be a stable oriented surface. Then S admits a hyperbolic metric such that ∂S (if nonempty) is a union of closed geodesics. Moreover, the operation which assigns to each hyperbolic metric on S its corresponding complex structure (rotation by $90°$ in the tangent planes) is bijective. It restricts to a bijection between hyperbolic metrics of finite area and complex structures of finite type.*

Definition 1.21 If j is a complex structure of finite type then the corresponding hyperbolic metric of finite area is called the *Poincaré metric* of (S, j).

We will prove Theorem 1.20 later on in the lecture. It implies that in the case of a stable surface S we may identify the space of all complex structures (of finite type) on S with the space of all hyperbolic metrics (with finite area) on S, both modulo orientation preserving diffeomorphisms. It will turn out that this point of view will be the most useful for us. The following lemma will be helpful for the proof of Theorem 1.20:

[1] If $\partial W \neq \emptyset$ then these two notions of completeness are not equivalent anymore (consider for example the closed unit disk in $\mathbb{R}^2$ with the Euclidean metric). In this case we say that W is complete if (W, ϱ) is complete as a metric space.

Lemma 1.22 *Let (S, j) be an oriented Riemann surface without boundary, and let h_1, h_2 be two complete Riemannian metrics on S with constant sectional curvature -1 such that they both induce the complex structure j (i.e. j rotates vectors in each tangent plane by $90°$ with respect to either metric). Then $h_1 \equiv h_2$.*

Proof Let $\pi : \hat{S} \to S$ be the universal cover of S, and denote the induced metrics on $\hat{S}$ by $\hat{h}_1$ and $\hat{h}_2$ which are also complete metrics on $\hat{S}$ of constant sectional curvature -1. By Theorem 1.15 both $(\hat{S}, \hat{h}_1)$ and $(\hat{S}, \hat{h}_2)$ are isometric to the hyperbolic plane. Composing these isometries with π we obtain covering maps

$$\pi_1 : (H^+, g_{H^+}) \to (S, h_1) \quad \text{and} \quad \pi_2 : (H^+, g_{H^+}) \to (S, h_2)$$

which are local isometries. They are also holomorphic with respect to the complex structure j on S and the standard complex structure i on H^+ (recall that j is induced by h_1, h_2. Then the complex structure on H^+ which makes π_1, π_2 holomorphic is the one induced by g_{H^+} which is the standard one). Let

$$\varsigma : (S, h_1) \to (S, h_2), \quad \varsigma(x) := x$$

which is holomorphic. We claim that ς is an isometry as well. Indeed, let $p \in S$ and $p_1 \in \pi_1^{-1}(p)$, $p_2 \in \pi_2^{-1}(p)$. The map $\varsigma \circ \pi_1 : (H^+, g_{H^+}) \to (S, h_2)$ is holomorphic, and it has a unique holomorphic lift

$$\phi : (H^+, i) \to (H^+, i), \quad \pi_2 \circ \phi = \varsigma \circ \pi_1$$

into the cover π_2 such that $\phi(p_1) = p_2$.

$$\begin{array}{ccc} (H^+, g_{H^+}) & \xrightarrow{\phi} & (H^+, g_{H^+}) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ (S, h_1) & \xrightarrow{\varsigma} & (S, h_2) \end{array}$$

Similarly, the map $\varsigma^{-1} \circ \pi_2 : (H^+, g_{H^+}) \to (S, h_1)$ has a unique lift

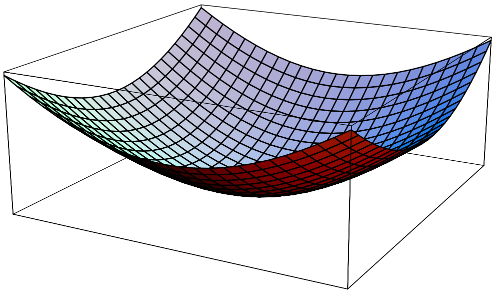
$$\psi : (H^+, g_{H^+}) \to (H^+, g_{H^+}), \quad \pi_1 \circ \psi = \varsigma^{-1} \circ \pi_2$$

into the cover π_1 such that $\psi(p_2) = p_1$. We have the following commutative diagram:

$$\begin{array}{ccc} & & H^+ \\ & \nearrow_{\psi \circ \phi} & \downarrow{\scriptstyle \pi_1} \\ H^+ & \xrightarrow[\pi_1]{} & S \end{array}$$

By the unique lifting property we conclude that $\psi \circ \phi = \mathrm{Id}_{H^+}$. Arguing in the same way for $\phi \circ \psi$ we finally get $\phi \in \mathrm{Conf}(\mathbb{H}) = I$, and the identity map $\varsigma : (S, h_1) \to (S, h_2)$ is then also an isometry as claimed. Then $h_2(x) = (\varsigma^{-1})^* h_1(x) = h_1(x)$. □

Fig. 1.4 The hyperbolic plane $H^2 \subset \mathbb{R}^3$



1.2.2 *The Hyperbolic Plane*

In order to understand hyperbolic metrics on Riemann surfaces we need some facts from elementary hyperbolic geometry first. We start by discussing the hyperbolic plane (following [28, 46, 74]). Consider the upper half plane $H^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ with the metric $g_{H^+}(z) := (\mathrm{Im}(z))^{-2} g_{\mathrm{eucl}}$ where g_{eucl} is the Euclidean metric. The space (H^+, g_{H^+}) is called the *upper half plane model of the hyperbolic plane*. We will investigate its properties. We first make some comments about the term 'upper half plane model'.

We consider on $\mathbb{R}^3$ the quadratic form

$$q(x_0, x_1, x_2) := -x_0^2 + x_1^2 + x_2^2.$$

Then the hyperbolic plane is defined by (Fig. 1.4)

$$\begin{aligned} H^2 &:= \left\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 > 0,\ q(x_0, x_1, x_2) = -1\right\} \\ &= \left\{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 = \sqrt{1 + x_1^2 + x_2^2}\right\}. \end{aligned}$$

The Lorentz metric $-dx_0^2 + dx_1^2 + dx_2^2$ induces a complete Riemannian metric g on H^2 which has constant sectional curvature -1 (we will verify this below). It is given by

$$g = \frac{1 + x_2^2}{x_0^2}\, dx_1^2 + \frac{1 + x_1^2}{x_0^2}\, dx_2^2 - \frac{2x_1 x_2}{x_0^2}\, dx_1\, dx_2.$$

The above definition is also called the *hyperboloid model* of the hyperbolic plane.

There are different ways to represent H^2. Not surprisingly, one of them will be the upper half plane model. Let $s = (-1, 0, 0)$ and define a diffeomorphism

$$f : H^2 \longrightarrow D = \left\{x \in \mathbb{R}^3 \mid x_0 = 0,\ x_1^2 + x_2^2 < 1\right\}$$

by stereographic projection, i.e.

$$f(x) = f(x_0, x_1, x_2) := \left(0, \frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}\right) = s - 2\frac{x - s}{q(x - s)}.$$