

# A Mathematical Introduction to Logic

*Second Edition*

Herbert B. Enderton

University of California, Los Angeles



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*for Eric and Bert*

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# Preface

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**T**his book, like the first edition, presents the basic concepts and results of logic: the topics are proofs, truth, and computability. As before, the presentation is directed toward the reader with some mathematical background and interests. In this revised edition, in addition to numerous “local” changes, there are three “global” ways in which the presentation has been changed:

First, I have attempted to make the material more *accessible* to the typical undergraduate student. In the main development, I have tried not to take for granted information or insights that might be unavailable to a junior-level mathematics student.

Second, for the instructor who wants to fit the book to his or her course, the organization has been made more *flexible*. Footnotes at the beginning of many of the sections indicate optional paths the instructor — or the independent reader — might choose to take.

Third, theoretical *computer science* has influenced logic in recent years, and some of that influence is reflected in this edition. Issues of computability are taken more seriously. Some material on finite models has been incorporated into the text.

The book is intended to serve as a textbook for an introductory mathematics course in logic at the junior-senior level. The objectives are to present the important concepts and theorems of logic and to explain their significance and their relationship to the reader’s other mathematical work.

As a text, the book can be used in courses anywhere from a quarter to a year in length. In one quarter, I generally reach the material on models of first-order theories (Section 2.6). The extra time afforded by a semester would permit some glimpse of undecidability, as in Section 3.0. In a second

term, the material of Chapter 3 (on undecidability) can be more adequately covered.

The book is intended for the reader who has not studied logic previously, but who has some experience in mathematical reasoning. There are no specific prerequisites aside from a willingness to function at a certain level of abstraction and rigor. There is the inevitable use of basic set theory. Chapter 0 gives a concise summary of the set theory used. One should not begin the book by studying this chapter; it is instead intended for reference if and when the need arises. The instructor can adjust the amount of set theory employed; for example it is possible to avoid cardinal numbers completely (at the cost of losing some theorems). The book contains some examples drawn from abstract algebra. But they are just examples, and are not essential to the exposition. The later chapters (Chapter 3 and 4) tend to be more demanding of the reader than are the earlier chapters.

Induction and recursion are given a more extensive discussion (in Section 1.4) than has been customary. I prefer to give an informal account of these subjects in lectures and have a precise version in the book rather than to have the situation reversed.

Exercises are given at the end of nearly all the sections. If the exercise bears a boldface numeral, then the results of that exercise are used in the exposition in the text. Unusually challenging exercises are marked with an asterisk.

I cheerfully acknowledge my debt to my teachers, a category in which I include also those who have been my colleagues or students. I would be pleased to receive comments and corrections from the users of this book. The Web site for the book can be found at <http://www.math.ucla.edu/~hbe/amil>.

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# Introduction

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**S**ymbolic logic is a mathematical model of deductive thought. Or at least that was true originally; as with other branches of mathematics it has grown beyond the circumstances of its birth. Symbolic logic is a model in much the same way that modern probability theory is a model for situations involving chance and uncertainty.

How are models constructed? You begin with a real-life object, for example an airplane. Then you select some features of this original object to be represented in the model, for example its shape, and others to be ignored, for example its size. And then you build an object that is like the original in some ways (which you call essential) and unlike it in others (which you call irrelevant). Whether or not the resulting model meets its intended purpose will depend largely on the selection of the properties of the original object to be represented in the model.

Logic is more abstract than airplanes. The real-life objects are certain "logically correct" deductions. For example,

All men are mortal.  
Socrates is a man.  
Therefore, Socrates is mortal.

The validity of inferring the third sentence (the conclusion) from the first two (the assumptions) does not depend on special idiosyncrasies of Socrates. The inference is justified by the form of the sentences rather than by empirical facts about mortality. It is not really important here what "mortal" means; it does matter what "all" means.

Borogoves are mimsy whenever it is brillig.  
It is now brillig, and this thing is a borogove.  
Hence this thing is mimsy.



Again we can recognize that the third sentence follows from the first two, even without the slightest idea of what a nimby borogove might look like.

Logically correct deductions are of more interest than the above frivolous examples might suggest. In fact, axiomatic mathematics consists of many such deductions laid end to end. These deductions made by the working mathematician constitute real-life originals whose features are to be mirrored in our model.

The logical correctness of these deductions is due to their form but is independent of their content. This criterion is vague, but it is just this sort of vagueness that prompts us to turn to mathematical models. A major goal will be to give, within a model, a precise version of this criterion. The questions (about our model) we will initially be most concerned with are these:

1. What does it mean for one sentence to "follow logically" from certain others?
2. If a sentence does follow logically from certain others, what methods of *proof* might be necessary to establish this fact?
3. Is there a gap between what we can *prove* in an axiomatic system (say for the natural numbers) and what is *true* about the natural numbers?
4. What is the connection between logic and computability?

Actually we will present two models. The first (sentential logic) will be very simple and will be woefully inadequate for interesting deductions. Its inadequacy stems from the fact that it preserves only some crude properties of real-life deductions. The second model (first-order logic) is admirably suited to deductions encountered in mathematics. When a working mathematician asserts that a particular sentence follows from the axioms of set theory, he or she means that this deduction can be translated to one in our model.

This emphasis on mathematics has guided the choice of topics to include. This book does not venture into many-valued logic, modal logic, or intuitionistic logic, which represent different selections of properties of real-life deductions.

Thus far we have avoided giving away much information about what our model, first-order logic, is like. As brief hints, we now give some examples of the expressiveness of its formal language. First, take the English sentence that asserts the set theoretic principle of extensionality, "If the same things are members of a first object as are members of a second object, then those objects are the same." This can be translated into our first-order language as

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

As a second example, we can translate the sentence familiar to calculus students, "For every positive number  $\epsilon$  there is a positive number  $\delta$  such that for any  $x$  whose distance from  $a$  is less than  $\delta$ , the distance between  $f(x)$  and  $b$  is less than  $\epsilon$ " as

$$\forall \epsilon (\epsilon > 0 \rightarrow \exists \delta (\delta > 0 \wedge \forall x (d_x a < \delta \rightarrow d_f x b < \epsilon)).$$

We have given some hints as to what we intend to do in this book. We should also correct some possible misimpressions by saying what we are not going to do. This book does not propose to teach the reader how to think. The word "logic" is sometimes used to refer to remedial thinking, but not by us. The reader already knows how to think. Here are some intriguing concepts to think about.



# Chapter ZERO

## Useful Facts about Sets

**W**e assume that the reader already has some familiarity with normal everyday set-theoretic apparatus. Nonetheless, we give here a brief summary of facts from set theory we will need; this will at least serve to establish the notation. It is suggested that the reader, instead of poring over this chapter at the outset, simply refer to it if and when issues of a set-theoretic nature arise in later chapters. The author's favorite book on set theory is of course his *Elements of Set Theory* (see the list of references at the end of this book).

First a word about argon. Throughout the book we will utilize an assortment of standard mathematical abbreviations. We use “ $\square$ ” to signify the end of a proof. A sentence “If ..., then ...” will sometimes be abbreviated “...  $\implies$  ...”. We also have “ $\Leftarrow$ ” for the converse implication (for the peculiar way the word “implication” is used in mathematics). For “if and only if” we use the shorter “iff” (this has become part of the mathematical language) and the symbol “ $\Leftrightarrow$ ”. For the word “therefore” we have the “ $\therefore$ ” abbreviation.

The notational device that extracts “ $x \neq y$ ” as the denial of “ $x = y$ ” and “ $x \notin y$ ” as the denial of “ $x \in y$ ” will be extended to other cases. For example, in Section 1.2 we define “ $\Sigma \models \pi$ ”; then “ $\Sigma \not\models \pi$ ” is its denial.

Now then, a *set* is a collection of things, called its members or elements. As usual, we write “ $t \in A$ ” to say that  $t$  is a member of  $A$ , and “ $t \notin A$ ” to say that  $t$  is not a member of  $A$ . We write “ $x = y$ ” to

mean that  $x$  and  $y$  are the same object. That is, the expression " $x$ " on the left of the equals sign is a name for the same object as is named by the other expression " $y$ ." If  $A = B$ , then for any object  $t$  it is automatically true that  $t \in A$  iff  $t \in B$ . This holds simply because  $A$  and  $B$  are the same thing. The converse is the principle of extensionality: If  $A$  and  $B$  are sets such that for every object  $t$ ,

$$t \in A \quad \text{iff} \quad t \in B,$$

then  $A = B$ . This reflects the idea of what a set is: a set is determined just by its members.

A useful operation is that of adjoining one extra object to a set. For a set  $A$ , let  $A; t$  be the set whose members are (i) the members of  $A$ , plus (ii) the (possibly new) member  $t$ . Here  $t$  may or may not already belong to  $A$ , and we have

$$A; t = A \cup \{t\}$$

using notation defined later, and

$$t \in A \quad \text{iff} \quad A; t = A.$$

One special set is the empty set  $\emptyset$ , which has no members at all. Any other set is said to be *nonempty*. For any object  $x$  there is the singleton set  $\{x\}$  whose only member is  $x$ . More generally, for any finite number  $x_1, \dots, x_n$  of objects there is the set  $\{x_1, \dots, x_n\}$  whose members are exactly those objects. Observe that  $\{x, y\} = \{y, x\}$ , as both sets have exactly the same members. We have only used different expressions to denote the set. If order matters, we can use ordered pairs (discussed later).

This notation will be stretched to cover some simple infinite cases. For example,  $\{0, 1, 2, \dots\}$  is the set  $\mathbb{N}$  of natural numbers, and  $\{\dots -2, -1, 0, 1, 2, \dots\}$  is the set  $\mathbb{Z}$  of all integers.

We write " $\{x \mid \_x \_ \}$ " for the set of all objects  $x$  such that  $\_x \_$ . We will take considerable liberty with this notation. For example,  $\{(m, n) \mid m < n \text{ in } \mathbb{N}\}$  is the set of all ordered pairs of natural numbers for which the first component is smaller than the second. And  $\{x \in A \mid \_x \_ \}$  is the set of all elements  $x$  in  $A$  such that  $\_x \_$ .

If  $A$  is a set all of whose members are also members of  $B$ , then  $A$  is a subset of  $B$ , abbreviated " $A \subseteq B$ ." Note that any set is a subset of itself. Also,  $\emptyset$  is a subset of every set. (" $\emptyset \subseteq A$ " is "vacuously true," since the task of verifying, for every member of  $\emptyset$ , that it also belongs to  $A$  requires doing nothing at all. Or from another point of view, " $A \subseteq B$ " can be false only if some member of  $A$  fails to belong to  $B$ . If  $A = \emptyset$ , this is impossible.) From the set  $A$  we can form a new set, the *power set*  $\mathcal{P}A$  of  $A$ , whose members are the subsets of  $A$ . Thus

$$\mathcal{P}A = \{x \mid x \subseteq A\}.$$

For example,

$$\mathcal{P}Z = \{\emptyset\},$$

$$\mathcal{P}\{Z\} = \{\emptyset, \{Z\}\}.$$

The *union* of  $A$  and  $B$ ,  $A \cup B$ , is the set of all things that are members of  $A$  or  $B$  (or both). For example,  $A; t = A \cup \{t\}$ . Similarly, the *intersection* of  $A$  and  $B$ ,  $A \cap B$ , is the set of all things that are members of both  $A$  and  $B$ . Sets  $A$  and  $B$  are *disjoint* iff their intersection is empty (i.e., if they have no members in common). A collection of sets is *pairwise disjoint* iff any two members of the collection are disjoint.

More generally, consider a set  $A$  whose members are themselves sets. The union,  $\bigcup A$ , of  $A$  is the set obtained by dumping all the members of  $A$  into a single set:

$$\bigcup A = \{x \mid x \text{ belongs to some member of } A\}.$$

Similarly for nonempty  $A$ ,

$$\bigcap A = \{x \mid x \text{ belongs to all members of } A\}.$$

For example, if

$$A = \{\{0, 1, 5\}, \{1, 6\}, \{1, 5\}\},$$

then

$$\bigcup A = \{0, 1, 5, 6\},$$

$$\bigcap A = \{1\}.$$

Two other examples are

$$A \cup B = \bigcup \{A, B\},$$

$$\bigcup \mathcal{P}A = A.$$

In cases where we have a set  $A_n$  for each natural number  $n$ , the union of all these sets,  $\bigcup \{A_n \mid n \in \mathbb{N}\}$ , is usually denoted " $\bigcup_{n \in \mathbb{N}} A_n$ " or just " $\bigcup_n A_n$ ."

The ordered pair  $\langle x, y \rangle$  of objects  $x$  and  $y$  must be defined in such a way that

$$\langle x, y \rangle = \langle u, v \rangle \quad \text{iff} \quad x = u \quad \text{and} \quad y = v.$$

Any definition that has this property will do; the standard one is

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}.$$

For ordered triples we define

$$\langle x, y, z \rangle = \langle \langle x, y \rangle, z \rangle.$$

More generally we define  $n$ -tuples recursively by

$$\langle x_1, \dots, x_{n+1} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{n+1} \rangle$$

for  $n > 1$ . It is convenient to define also  $\langle x \rangle = x$ ; the preceding equation then holds also for  $n = 1$ .  $S$  is a *finite sequence* (or *string*) of members of  $A$  iff for some positive integer  $n$ , we have  $S = \langle x_1, \dots, x_n \rangle$ , where each  $x_i \in A$ . (Finite sequences are often defined to be certain finite functions, but the above definition is slightly more convenient for us.)

A *segment* of the finite sequence  $S = \langle x_1, \dots, x_n \rangle$  is a finite sequence

$$\langle x_k, x_{k+1}, \dots, x_{m-1}, x_m \rangle, \quad \text{where } 1 \leq k \leq m \leq n.$$

This segment is an *initial segment* iff  $k = 1$  and it is *proper* iff it is different from  $S$ .

If  $\langle x_1, \dots, x_n \rangle = \langle y_1, \dots, y_n \rangle$ , then it is easy to see that  $x_i = y_i$  for  $1 \leq i \leq n$ . (The proof uses induction on  $n$  and the basic property of ordered pairs.) But if  $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_n \rangle$ , then it does not in general follow that  $m = n$ . After all, every ordered triple is also an ordered pair. But we claim that  $m$  and  $n$  can be unequal only if some  $x_i$  is itself a finite sequence of  $y_j$ 's, or the other way around:

**LEMMA 0A** Assume that  $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_n, \dots, y_{n+k} \rangle$ . Then  $x_1 = \langle y_1, \dots, y_{k+1} \rangle$ .

**PROOF.** We use induction on  $m$ . If  $m = 1$ , the conclusion is immediate. For the inductive step, assume that  $\langle x_1, \dots, x_m, x_{m+1} \rangle = \langle y_1, \dots, y_n, y_{n+1}, y_{n+1+k} \rangle$ . Then the first components of this ordered pair must be equal:  $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_{n+k} \rangle$ . Now apply the inductive hypothesis.  $\square$

For example, suppose that  $A$  is a set such that no member of  $A$  is a finite sequence of other members. Then if  $\langle x_1, \dots, x_m \rangle = \langle y_1, \dots, y_n \rangle$  and each  $x_i$  and  $y_j$  is in  $A$ , then by the above lemma  $m = n$ . Whereupon we have  $x_i = y_i$  as well.

From sets  $A$  and  $B$  we can form their *Cartesian product*, the set  $A \times B$  of all pairs  $\langle x, y \rangle$  for which  $x \in A$  and  $y \in B$ .  $A^n$  is the set of all  $n$ -tuples of members of  $A$ . For example,  $A^3 = (A \times A) \times A$ .

A *relation*  $R$  is a set of ordered pairs. For example, the ordering relation on the numbers 0-3 is captured by  $<$  and in fact is  $<$  — the set of ordered pairs

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

The *domain* of  $R$  (written  $\text{dom } R$ ) is the set of all objects  $x$  such that  $\langle x, y \rangle \in R$  for some  $y$ . The *range* of  $R$  (written  $\text{ran } R$ ) is the set of all objects  $y$  such that  $\langle x, y \rangle \in R$  for some  $x$ . The union of  $\text{dom } R$  and  $\text{ran } R$  is the *field* of  $R$ ,  $\text{fld } R$ .

An  $n$ -ary relation on  $A$  is a subset of  $A^n$ . If  $n > 1$ , it is a relation. But a 1-ary (unary) relation on  $A$  is simply a subset of  $A$ . A particularly simple binary relation on  $A$  is the equality relation  $\{(x, x) \mid x \in A\}$  on  $A$ . For an  $n$ -ary relation  $R$  on  $A$  and subset  $B$  of  $A$ , the *restriction* of  $R$  to  $B$  is the intersection  $R \cap B^n$ . For example, the relation displayed above is the restriction to the set  $B = \{0, 1, 2, 3\}$  of the ordering relation on  $\mathbb{N}$ .

A *function* is a relation  $F$  with the property of being *single-valued*: For each  $x$  in  $\text{dom } F$  there is only one  $y$  such that  $(x, y) \in F$ . As usual, this unique  $y$  is said to be the value  $F(x)$  that  $F$  assumes at  $x$ . (This notation goes back to Euler. It is a pity he did not choose  $(x)F$  instead; that would have been helpful for the *composition* of functions;  $f \circ g$  is the function whose value at  $x$  is  $f(g(x))$ , obtained by applying first  $g$  and then  $f$ .)

We say that  $F$  maps  $A$  into  $B$  and write

$$F : A \rightarrow B$$

to mean that  $F$  is a function,  $\text{dom } F = A$ , and  $\text{ran } F \subseteq B$ . If in addition  $\text{ran } F = B$ , then  $F$  maps  $A$  onto  $B$ .  $F$  is *one-to-one* iff for each  $y$  in  $\text{ran } F$  there is only one  $x$  such that  $(x, y) \in F$ . If the pair  $(x, y)$  is in  $\text{dom } F$ , then we let  $F(x, y) = F(\{(x, y)\})$ . This notation is extended to  $n$ -tuples,  $F(x_1, \dots, x_n) = F(\{(x_1, \dots, x_n)\})$ .

An  $n$ -ary operation on  $A$  is a function mapping  $A^n$  into  $A$ . For example, addition is a binary operation on  $\mathbb{N}$ , whereas the successor operation  $S$  (where  $S(n) = n + 1$ ) is a unary operation on  $\mathbb{N}$ . If  $f$  is an  $n$ -ary operation on  $A$ , then the *restriction* of  $f$  to a subset  $B$  of  $A$  is the function  $g$  with domain  $B^n$  which agrees with  $f$  at each point of  $B^n$ . Thus,

$$g = f \cap (B^n \times A).$$

This  $g$  will be an  $n$ -ary operation on  $B$  iff  $B$  is *closed* under  $f$ , in the sense that  $f(b_1, \dots, b_n) \in B$  whenever each  $b_i$  is in  $B$ . In this case,  $g = f \cap B^{n-1}$ , in agreement with our definition of the restriction of a relation. For example, the addition operation on  $\mathbb{N}$ , which contains such triples as  $\{(3, 2), 5\}$ , is the restriction to  $\mathbb{N}$  of the addition operation on  $\mathbb{R}$ , which contains many more triples.

A particularly simple unary operation on  $A$  is the *identity* function  $Id$  on  $A$ , given by the equation

$$Id(x) = x \quad \text{for } x \in A.$$

Thus  $Id = \{(x, x) \mid x \in A\}$ .

For a relation  $R$ , we define the following:

$R$  is *reflexive* on  $A$  iff  $(x, x) \in R$  for every  $x$  in  $A$ .

$R$  is *symmetric* iff whenever  $(x, y) \in R$ , then also  $(y, x) \in R$ .

$R$  is *transitive* iff whenever both  $(x, y) \in R$  and  $(y, z) \in R$  (if this ever happens), then also  $(x, z) \in R$ .

$R$  satisfies *trichotomy* on  $A$  iff for every  $x$  and  $y$  in  $A$ , exactly one of the three possibilities,  $(x, y) \in R$ ,  $x = y$ , or  $(y, x) \in R$ , holds.

$R$  is an *equivalence relation* on  $A$  iff  $R$  is a binary relation on  $A$  that is reflexive on  $A$ , symmetric, and transitive.

$R$  is an *ordering relation* on  $A$  iff  $R$  is transitive and satisfies trichotomy on  $A$ .

For an equivalence relation  $R$  on  $A$  we define, for  $x \in A$ , the *equivalence class*  $[x]$  of  $x$  to be  $\{y \mid (x, y) \in R\}$ . The equivalence classes then *partition*  $A$ . That is, the equivalence classes are subsets of  $A$  such that each member of  $A$  belongs to exactly one equivalence class. For  $x$  and  $y$  in  $A$ ,

$$[x] = [y] \quad \text{iff} \quad (x, y) \in R.$$

The set  $\mathbb{N}$  of natural numbers is the set  $\{0, 1, 2, \dots\}$ . (Natural numbers can also be defined set-theoretically, a point that arises briefly in Section 3.7.) A set  $A$  is *finite* iff there is some one-to-one function  $f$  mapping (for some natural number  $n$ ) the set  $A$  onto  $\{0, 1, \dots, n-1\}$ . (We can think of  $f$  as "counting" the members of  $A$ .)

A set  $A$  is *countable* iff there is some function mapping  $A$  one-to-one into  $\mathbb{N}$ . For example, any finite set is obviously countable. Now consider an infinite countable set  $A$ . Then from the given function  $f$  mapping  $A$  one-to-one into  $\mathbb{N}$ , we can extract a function  $f'$  mapping  $A$  one-to-one onto  $\mathbb{N}$ . For some  $a_0 \in A$ ,  $f(a_0)$  is the least member of  $\text{ran } f$ , let  $f'(a_0) = 0$ . In general there is a unique  $a_n \in A$  such that  $f(a_n)$  is the  $(n+1)$ st member of  $\text{ran } f$ ; let  $f'(a_n) = n$ . Note that  $A = \{a_0, a_1, \dots\}$ . (We can also think of  $f'$  as "counting" the members of  $A$ , only now the counting process is infinite.)

**THEOREM 0B** Let  $A$  be a countable set. Then the set of all finite sequences of members of  $A$  is also countable.

**PROOF.** The set  $S$  of all such finite sequences can be characterized by the equation

$$S = \bigcup_{n \in \mathbb{N}} A^{n+1}.$$

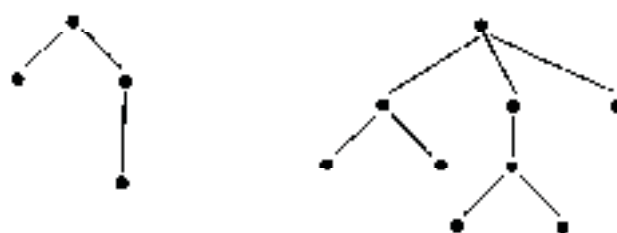
Since  $A$  is countable, we have a function  $f$  mapping  $A$  one-to-one into  $\mathbb{N}$ .

The basic idea is to map  $S$  one-to-one into  $\mathbb{N}$  by assigning to  $\langle a_0, a_1, \dots, a_m \rangle$  the number  $2^{f(a_0)+1} 3^{f(a_1)+1} \dots p_m^{f(a_m)+1}$ , where  $p_m$  is the  $(m+1)$ st prime. This suffers from the defect that this assignment might not be well-defined. For conceivably there could be  $\langle a_0, a_1, \dots, a_m \rangle = \langle b_0, b_1, \dots, b_n \rangle$ , with  $a_i$  and  $b_j$  in  $A$  but with  $m \neq n$ . But this is not serious: just assign to each member of  $S$  the *smallest* number obtainable in the above

fashion. This gives us a well-defined map; it is easy to see that it is one-to-one.

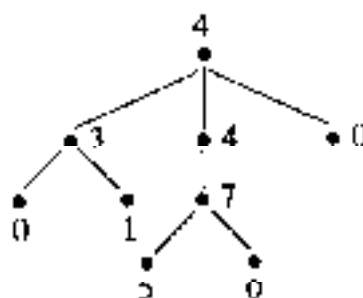
At times we will speak of *trees*, which can be useful in providing intuitive pictures of some situations. But our comments on trees will always be informal; the theorems and proofs will not rely on trees. Accordingly, our discussion here of trees will be informal.

For each tree there is an underlying finite partial ordering. We can draw a picture of this partial ordering  $R$ ; if  $(a, b) \in R$ , then we put  $a$  lower than  $b$  and connect the points by a line. Pictures of two typical tree orderings are shown.



(In mathematics, trees grow downward, not upward.) There is always a highest point in the picture (the *root*). Furthermore, while branching is permitted below some vertex, the points *above* any given vertex must lie along a line.

In addition to this underlying finite partial ordering, a tree also has a labeling function whose domain is the set of vertices. For example, one tree, in which the labels are natural numbers, is shown.



At a few points in the book we will use the axiom of choice. But usually these uses can be eliminated if the theorems in question are restricted to countable languages. Of the many equivalent statements of the axiom of choice, Zorn's lemma is especially useful.

Say that a collection  $C$  of sets is a *chain* iff for any elements  $x$  and  $y$  of  $C$ , either  $x \subseteq y$  or  $y \subseteq x$ .

**ZORN'S LEMMA** Let  $A$  be a set such that for any chain  $C \subseteq A$ , the set  $\bigcup C$  is in  $A$ . Then there is some element  $m \in A$  which is maximal in the sense that it is not a subset of any other element of  $A$ .



## Cardinal Numbers

All infinite sets are big, but some are bigger than others. (For example, the set of real numbers is bigger than the set of integers.) Cardinal numbers provide a convenient, although not indispensable, way of talking about the size of sets.

It is natural to say that two sets  $A$  and  $B$  have the same size iff there is a function that maps  $A$  one-to-one onto  $B$ . If  $A$  and  $B$  are finite, then this concept is equivalent to the usual one: If you count the members of  $A$  and the members of  $B$ , then you get the same number both times. But it is applicable even to infinite sets  $A$  and  $B$ , where counting is difficult.

Formally, then, say that  $A$  and  $B$  are *equinumerous* (written  $A \sim B$ ) iff there is a one-to-one function mapping  $A$  onto  $B$ . For example, the set  $\mathbb{N}$  of natural numbers and the set  $\mathbb{Z}$  of integers are equinumerous. It is easy to see that equinumerosity is reflexive, symmetric, and transitive.

For finite sets we can use natural numbers as measures of size. The same natural number would be assigned to two finite sets (as measures of their size) iff the sets were equinumerous. Cardinal numbers are introduced to enable us to generalize this situation to infinite sets.

To each set  $A$  we can assign a certain object, the *cardinal number* (or *cardinality*) of  $A$  (written  $\text{card } A$ ), in such a way that two sets are assigned the same cardinality iff they are equinumerous:

$$\text{card } A = \text{card } B \quad \text{iff} \quad A \sim B. \quad (\mathbf{K})$$

There are several ways of accomplishing this: the standard one these days takes  $\text{card } A$  to be the least ordinal equinumerous with  $A$ . (The success of this definition relies on the axiom of choice.) We will not discuss ordinals here, since for our purposes it matters very little what  $\text{card } A$  actually is, any more than it matters what the number  $\lambda$  actually is. What matters most is that  $(\mathbf{K})$  holds. It is helpful, however, if for a finite set  $A$ ,  $\text{card } A$  is the natural number telling how many elements  $A$  has. Something is a *cardinal number*, or simply a *cardinal*, iff it is  $\text{card } A$  for some set  $A$ .

(Georg Cantor, who first introduced the concept of cardinal number, characterized in 1895 the cardinal number of a set  $M$  as "the general concept which, with the help of our active intelligence, comes from the set  $M$  upon abstraction from the nature of its various elements and from the order of their being given.")

Say that  $A$  is *dominated* by  $B$  (written  $A \preceq B$ ) iff  $A$  is equinumerous with a subset of  $B$ . In other words,  $A \preceq B$  iff there is a one-to-one function mapping  $A$  into  $B$ . The companion concept for cardinals is

$$\text{card } A \preceq \text{card } B \quad \text{iff} \quad A \preceq B.$$

(It is easy to see that  $\preceq$  is well defined; that is, whether or not  $\kappa \preceq \lambda$  depends only on the cardinals  $\kappa$  and  $\lambda$  themselves, and not the choice of

sets having these cardinalities.) Dominance is reflexive and transitive. A set  $A$  is dominated by  $\mathbb{N}$  iff  $A$  is countable. The following is a standard result in this subject.

**SCHRÖDER-BERNSTEIN THEOREM** (a) For any sets  $A$  and  $B$ , if  $A \preceq B$  and  $B \preceq A$ , then  $A \sim B$   
 (b) For any cardinal numbers  $\kappa$  and  $\lambda$ , if  $\kappa \leq \lambda$  and  $\lambda \leq \kappa$ , then  $\kappa = \lambda$ .

Part (b) is a simple restatement of part (a) in terms of cardinal numbers. The following theorem, which happens to be equivalent to the axiom of choice, is stated in the same dual manner.

**THEOREM OC** (a) For any sets  $A$  and  $B$ , either  $A \preceq B$  or  $B \preceq A$ .  
 (b) For any cardinal numbers  $\kappa$  and  $\lambda$ , either  $\kappa < \lambda$  or  $\lambda < \kappa$ .

Thus of any two cardinals, one is smaller than the other. (In fact, any nonempty set of cardinal numbers contains a smallest member.) The smallest cardinals are those of finite sets:  $0, 1, 2, \dots$ . There is next the smallest infinite cardinal,  $\text{card } \mathbb{N}$ , which is given the name  $\aleph_0$ . Thus we have

$$0, 1, 2, \dots, \aleph_0, \aleph_1, \dots,$$

where  $\aleph_1$  is the smallest cardinal larger than  $\aleph_0$ . The cardinality of the real numbers,  $\text{card } \mathbb{R}$ , is called " $2^{\aleph_0}$ ." Since  $\mathbb{R}$  is uncountable, we have  $\aleph_0 < 2^{\aleph_0}$ .

The operations of addition and multiplication, long familiar for finite cardinals, can be extended to all cardinals. To compute  $\kappa + \lambda$  we choose disjoint sets  $A$  and  $B$  of cardinality  $\kappa$  and  $\lambda$ , respectively. Then

$$\kappa + \lambda = \text{card}(A \cup B).$$

This is well defined; i.e.,  $\kappa + \lambda$  depends only on  $\kappa$  and  $\lambda$ , and not on the choice of the disjoint sets  $A$  and  $B$ . For multiplication we use

$$\kappa \cdot \lambda = \text{card}(A \times B).$$

Clearly these definitions are correct for finite cardinals. The arithmetic of infinite cardinals is surprisingly simple (with the axiom of choice). The sum or product of two infinite cardinals is simply the larger of them:

**CARDINAL ARITHMETIC THEOREM** For cardinal numbers  $\kappa$  and  $\lambda$ , if  $\kappa \leq \lambda$  and  $\lambda$  is infinite, then  $\kappa + \lambda = \lambda$ . Furthermore, if  $\kappa \neq 0$ , then  $\kappa \cdot \lambda = \lambda$ .

In particular, for infinite cardinals  $\kappa$ ,

$$\aleph_0 \cdot \kappa = \kappa.$$

**THEOREM 0D.** For an infinite set  $A$ , the set  $\bigcup_n A^{n+1}$  of all finite sequences of elements of  $A$  has cardinality equal to  $\text{card } A$ .

We already proved this for the case of a countable  $A$  (see Theorem 0B).

**PROOF.** Each  $A^{n+1}$  has cardinality equal to  $\text{card } A$ , by the cardinal arithmetic theorem (applied  $n$  times). So we have the union of  $\aleph_0$  sets of this size, yielding  $\aleph_0 \cdot \text{card } A = \text{card } A$  points altogether.  $\dashv$

**EXAMPLE.** It follows that the set of algebraic numbers has cardinality  $\aleph_0$ . First, we can identify each polynomial (in one variable) over the integers with the sequence of its coefficients. Then by the theorem there are  $\aleph_0$  polynomials. Each polynomial has a finite number of roots. To give an extravagant upper bound, note that even if each polynomial had  $\aleph_0$  roots, we would then have  $\aleph_0 \cdot \aleph_0 = \aleph_0$  algebraic numbers altogether. Since there are at least this many, we are done.

Since there are uncountably many (in fact,  $2^{\aleph_0}$ ) real numbers, it follows that there are uncountably many (in fact,  $2^{\aleph_0}$ ) transcendental numbers.



# Chapter ONE

## Sentential Logic

### SECTION 1.0

#### Informal Remarks on Formal Languages

We are about to construct (in the next section) a language into which we can translate English sentences. Unlike natural languages (such as English or Chinese), it will be a formal language, with precise formation rules. But before the precision begins, we will consider here some of the features to be incorporated into the language.

As a first example, the English sentence "Traces of potassium were observed" can be translated into the formal language as, say, the symbol  $K$ . Then for the closely related sentence "Traces of potassium were not observed," we can use  $(\neg K)$ . Here  $\neg$  is our negation symbol, read as "not." One might also think of translating "Traces of potassium were not observed" by some new symbol, for example,  $J$ , but we will prefer instead to break such a sentence down into atomic parts as much as possible. For an unrelated sentence, "The sample contained chlorine," we choose, say, the symbol  $C$ . Then the following compound English sentences can be translated as the formulas shown at the right:

If traces of potassium were observed, then the sample did not contain chlorine.  $(K \rightarrow (\neg C))$

The sample contained chlorine, and traces of potassium were observed.  $(C \wedge K)$

The second case uses our conjunction symbol  $\wedge$  to translate "and." The first one uses the more familiar arrow to translate "if . . . then . . ." In the following example the disjunction symbol  $\vee$  is used to translate "or":

Either no traces of potassium were observed, or the sample did not contain chlorine.  $((\neg K) \vee (\neg C))$

Neither did the sample contain chlorine, nor were traces of potassium observed.  $(\neg (C \vee K))$   
or  
 $((\neg C) \wedge (\neg K))$

For this last sentence, we have two alternative translations. The relationship between them will be discussed later.

One important aspect of the decompositions we will make of compound sentences is that whenever we are given the truth or falsity of the atomic parts, we can then immediately calculate the truth or falsity of the compound. Suppose, for example, that the chemist emerges from her laboratory and announces that she observed traces of potassium but that the sample contained no chlorine. We then know that the four above sentences are true, false, true, and false, respectively. In fact, we can construct in advance a table giving the four possible experimental results (Table I). We will return to the discussion of such tables in Section 1.2.

TABLE I

K	C	$(\neg(C \vee K))$	$((\neg C) \wedge (\neg K))$
F	F	T	T
F	T	F	F
T	F	F	F
T	T	F	F

Use of formal languages will allow us to escape from the imprecision and ambiguities of natural languages. But this is not done without cost: our formal languages will have a sharply limited degree of expressiveness.

In order to describe a formal language we will generally give three pieces of information:

1. We will specify the set of symbols (the alphabet). In the present case of sentential logic some of the symbols are:

$$(\cdot), \rightarrow, \neg, A_1, A_2, \dots$$

2. We will specify the rules for forming the "grammatically correct" finite sequences of symbols. (Such sequences will be called *well-formed formulas* or *wff's*.) For example, in the present case

$$(A_1 \rightarrow (\neg A_2))$$

will be a wff, whereas

$$)) \rightarrow A_3$$

will not.

3. We will also indicate the allowable translations between English and the formal language. The symbols  $A_1, A_2, \dots$  can be translations of declarative English sentences.

Only in this third part is any meaning assigned to the wffs. This process of assigning meaning guides and motivates all we do. But it will also be observed that it would theoretically be possible to carry out various manipulations with wffs in complete ignorance of any possible meaning. A person aware of only the first two pieces of information listed above could perform some of the things we will do, but it would make no sense to him.

Before proceeding, let us look briefly at another class of formal languages of widespread interest today. These are the languages used by (or at least in connection with) digital computers.

There are many of these languages. In one of them a typical wff is

01101011010100011111000100001111010.

In another a typical wff is

STEP#ADDIMAX, A.

(Here # is a symbol called a blank; it is brought into the alphabet so that a wff will be a string of symbols.) A well-known language called C++ has wffs such as

while(\*s++);

In all cases there is a given way of translating the wffs into English, and (for a restricted class of English sentences) a way of translating from English into the formal language. But the computer is unaware of the English language. An unthinking automaton, the computer juggles symbols and follows its program slavishly. We could approach formal languages that way, too, but it would not be much fun.

## SECTION 1.1

### The Language of Sentential Logic

We assume we are given an infinite sequence of distinct objects which we will call symbols, and to which we now give names (Table II). We further assume that no one of these symbols is a finite sequence of other symbols.

TABLE II

Symbol	Verbose name	Remarks
(	left parenthesis	punctuation
)	right parenthesis	punctuation
$\neg$	negation symbol	English: not
$\wedge$	conjunction symbol	English: and
$\vee$	disjunction symbol	English: or (inclusive)
$\rightarrow$	conditional symbol	English: if $\_$ , then $\_$
$\leftrightarrow$	biconditional symbol	English: if and only if
$A_1$	first sentence symbol	
$A_2$	second sentence symbol	
...		
$A_n$	$n$ th sentence symbol	
...		

Several remarks are now in order:

1. The five symbols

$$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

are called *sentential connective* symbols. Their use is suggested by the English translation given above. The sentential connective symbols, together with the parentheses, are the *logical symbols*. In translating to and from English, their role never changes. The sentence symbols are the *parameters* (or *nonlogical symbols*). Their translation is not fixed; instead they will be open to a variety of interpretations, as we shall illustrate shortly.

2. We have included infinitely many sentence symbols. On the one hand, a more modest alternative would be to have *one* sentence symbol  $A$ , and a prime ' . Then we could use the potentially infinite sequence

$$A, A', A'', \dots$$

in place of

$$A_1, A_2, A_3, \dots$$

This alternative has the advantage that it brings the total number of different symbols down to nine. On the other hand, a *less* modest alternative would be to allow an arbitrary set of ~~1~~ sentence symbols, countable or not. Much of what is said in this chapter would continue to be applicable in this case: the exceptions are primarily in Section 1.7.

3. Some logicians prefer to call  $A_n$  the  $n$ th *proposition* symbol (and to speak of propositional logic instead of sentential logic). This stems

from wanting the word "sentence" to refer to a particular utterance, and wanting a proposition to be that which a sentence asserts.

4. We call these objects "symbols," but we remain neutral as to what the exact ontological status of symbols might be. In the leftmost column of our list of symbols, names of the symbols are printed. For example,  $A_{243}$  is one symbol, namely the 243rd sentence symbol. (On the other hand, " $A_{243}$ " is a name of that symbol. The conditional symbol may or may not have the geometric property of being shaped like an arrow, although its name " $\rightarrow$ " does.) The symbols themselves can be sets, numbers, marbles, or objects from a universe of linguistic objects. In the last case, it is conceivable that they are actually the same things as the names we use for them. Another possibility, which will be explained in the next chapter, is that the sentence symbols are themselves formulas in another language.

5. We have assumed that no symbol is a finite sequence of other symbols. We mean by this that not only are the symbols listed distinct (e.g.,  $A_3 \neq \leftrightarrow$ ), but no one of them is a finite sequence of two or more symbols. For example, we demand that  $A_3 \neq (\neg, A_1, \cup)$ . The purpose of this assumption is to assure that finite sequences of symbols are uniquely decomposable. If

$$\langle a_1, \dots, a_m \rangle = \langle b_1, \dots, b_n \rangle$$

and each  $a_i$  and each  $b_j$  is a symbol, then  $m = n$  and  $a_i = b_i$ . (See Chapter 0, Lemma 0A, and subsequent remarks.)

An *expression* is a finite sequence of symbols. We can specify an expression by concatenating the names of the symbols; thus  $(\neg A_2)$  is the sequence  $((, \neg, A_1, ))$ . This notation is extended: If  $\alpha$  and  $\beta$  are sequences of symbols, then  $\alpha\beta$  is the sequence consisting first of the symbols in the sequence  $\alpha$  followed by the symbols in the sequence  $\beta$ .

For example, if  $\alpha$  and  $\beta$  are the expressions given by the equations

$$\alpha = (\neg A_1),$$

$$\beta = A_2,$$

then  $(\alpha \rightarrow \beta)$  is the expression

$$((\neg A_1) \rightarrow A_2).$$

We should now look at a few examples of possible translations of English sentences into expressions of the formal language. Let  $A, B, \dots, Z$  be the first twenty-six sentence symbols. (For example,  $E = A_5$ .)

1. English: The suspect must be released from custody. Translation:  $R$ .

English: The evidence obtained is admissible. Translation:  $E$ .

English: The evidence obtained is inadmissible. Translation:  $(\neg E)$ .



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